A SIMPLER PROOF OF TOROIDALIZATION OF MORPHISMS FROM 3-FOLDS TO SURFACES

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1. Introduction

Let $\mathfrak{k}$ be an algebraically closed field of characteristic zero. Toroidal varieties and morphisms of toroidal varieties over $\mathfrak{k}$ are defined in [32], [4] and [5]. If X is nonsingular, then the choice of a SNC divisor on X makes X into a toroidal variety.

Suppose that $\Phi : X \to Y$ is a dominant morphism of nonsingular $\mathfrak{k}$-varieties, and there is a SNC divisor D_Y on Y such that $D_X = \Phi^{-1}(D_Y)$ is a SNC divisor on X. Then Φ is torodial (with respect to D_Y and D_X) if and only if $\Phi^*(\Omega^1_Y(\log D_Y))$ is a subbundle of $\Omega^1_X(\log D_X)$ (Lemma 1.5 [15]). A toroidal morphism can be expressed locally by monomials. All of the cases are written down for toroidal morphisms from a 3-fold to a surface in Lemma 19.3 [15].

The toroidalization problem is to determine, given a dominant morphism $f : X \to Y$ of $\mathfrak{k}$-varieties, if there exists a commutative diagram

$$\begin{array}{ccc} X_1 & \overset{f_1}{\to} & Y_1 \\ \Phi\downarrow & & \downarrow\Psi \\ X & \overset{f}{\to} & Y \end{array}$$

such that Φ and Ψ are products of blow ups of nonsingular subvarieties, X_1 and Y_1 are nonsingular, and there exist SNC divisors D_{Y_1} on Y_1 and $D_{X_1} = f^*(D_{Y_1})$ on X_1 such that f_1 is toroidal (with respect to D_{X_1} and D_{Y_1}). This is stated in Problem 6.2.1 of [5]. Some papers where related problems are considered are [4] and [35].

The toroidalization problem does not have a positive answer in positive characteristic p, even for maps of curves; $t = x^p + x^{p+1}$ gives a simple example.

In characteristic zero, the toroidalization problem has an affirmative answer if Y is a curve and X has arbitrary dimension; this is really embedded resolution of hypersurface singularities, so follows from resolution of singularities ([27], and simplified proofs [7], [8], [18], [22], [23], [34] and [41]). There are several proofs for the case of maps of a surface to a surface (some references are [3], [20] and Corollary 6.2.3 [5]). The case of a morphism from a 3-fold to a surface is proven in [15], and the case of a morphism from a 3-fold to a 3-fold is proven in [16].

The problem of toroidalization is a resolution of singularities type problem. When the dimension of the base is larger than one, the problem shares many of the complexities of resolution of vector fields ([38], [9],[36]) and of resolution of singularities in positive characteristic (some references are [1], [2], [28], [10], [11], [12], [17], [21], [24], [25], [26], [29], [30], [31], [33], [39], [40], [6]). In particular, natural invariants do not have a "hypersurface of maximal contact" and are sometimes not upper semicontinuous.

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Toroidalization, locally along a fixed valuation, is proven in all dimensions and relative dimensions in [13] and [14].

The proof of toroidalization of a dominant morphism from a 3-fold to a surface given in [15] consists of 2 steps.

The first step is to prove "strong preparation". Suppose that X is a nonsingular variety, S is a nonsingular surface with a SNC divisor D_S, and $f : X \to S$ is a dominant morphism such that $D_X = f^{-1}(D_S)$ is a SNC divisor on X which contains the locus where f is not smooth. f is strongly prepared if $f^*(\Omega^2_S(\log D_S)) = \mathcal{I}\mathcal{M}$ where $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf, and $\mathcal{M}$ is a subbundle of $\Omega^2_X(\log D_X)$ (Lemma 1.7 [15]). A strongly prepared morphism has nice local forms which are close to being toroidal (page 7 of [15]).

Strong preparation is the construction of a commutative diagram

$$\begin{array}{ccc} X_1 & & \\ \downarrow & \searrow & \\ X & \xrightarrow{f} & S \end{array}$$

where S is a nonsingular surface with a SNC divisor D_S such that $D_X = f^*(D_S)$ is a SNC divisor on the nonsingular variety X which contains the locus where f is not smooth, the vertical arrow is a product of blow ups of nonsingular subvarieties so that $X_1 \to S$ is strongly prepared. Strong preparation of morphisms from 3-folds to surfaces is proven in Theorem 17.3 of [15].

The second step is to prove that a strongly prepared morphism from a 3-fold to a surface can be toroidalized. This is proven in Sections 18 and 19 of [15].

This second step is generalized in [19] to prove that a strongly prepared morphism from an n-fold to a surface can be toroidalized. Thus to prove toroidalization of a morphism from an n-fold to a surface, it suffices to proof strong preparation.

The proof of strong preparation in [15] is extremely complicated, and does not readily generalize to higher dimensions. The proof of this result occupies 170 pages of [15]. We mention that that the main invariant considered in this paper, ν, can be interpreted as the adopted order of Section 1.2 of [9] of the 2-form $du \wedge dv$.

In this paper, we give a significantly simpler and more conceptual proof of strong preparation of morphisms of 3-folds to surfaces. It is our hope that this proof can be extended to prove strong preparation for morphisms of n-folds to surfaces, for $n > 3$. The proof is built around a new upper semicontinuous invariant σ_D, whose value is a natural number or ∞. if $\sigma_D(p) = 0$ for all $p \in X$, then $X \to S$ is prepared (which is slightly stronger than being strongly prepared). A first step towards obtaining a reduction in σ_D is to make X 3-prepared, which is achieved in Section 3. This is a nicer local form, which is proved by making a local reduction to lower dimension. The proof proceeds by performing a toroidal morphism above X to obtain that X is 3-prepared at all points except for a finite number of 1-points. Then general curves through these points lying on D_X are blown up to achieve 3-preparation everywhere on X. if X is 3-prepared at a point p, then there exists an étale cover U_p of an affine neighborhood of p and a local toroidal structure $\overline{D}_p$ at p (which contains D_X) such that there exists a projective toroidal morphism $\Psi : U' \to U_p$ such that σ_D has dropped everywhere above p (Section 4). The final step of the proof is to make these local constructions algebraic, and to patch them. This is accomplished in Section 5. In Section 6 we state and prove strong preparation for morphisms of 3-folds to surfaces (Theorem 6.1) and toroidalization of morphisms from 3-folds to surfaces (Theorem 6.2).

2. The invariant σ_D, 1-preparation and 2-preparation.

For the duration of the paper, $\mathfrak{k}$ will be an algebraically closed field of characteristic zero. We will write curve (over $\mathfrak{k}$) to mean a 1-dimensional $\mathfrak{k}$-variety, and similarly for surfaces and 3-folds. We will assume that varieties are quasi-projective. This is not really a restriction, by the fact that after a sequence of blow ups of nonsingular subvarieties, all varieties satisfy this condition. By a general point of a $\mathfrak{k}$-variety Z, we will mean a member of a nontrivial open subset of Z on which some specified good condition holds.

A reduced divisor D on a nonsingular variety Z of dimension n is a simple normal crossings divisor (SNC divisor) if all irreducible components of D are nonsingular, and if $p \in Z$, then there exists a regular system of parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{Z,p}$ such that $x_1x_2\cdots x_r = 0$ is a local equation of D at p, where $r \le n$ is the number of irreducible components of D containing p. Two nonsingular subvarieties X and Y intersect transversally at $p \in X \cap Y$ if there exists a regular system of parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{Z,p}$ and subsets $I, J \subset \{1, \ldots, n\}$ such that $\mathcal{I}_X, p = (x_i \mid i \in I)$ and $\mathcal{I}_Y, p = (x_j \mid j \in J)$.

Definition 2.1. *Let S be a nonsingular surface over $\mathfrak{k}$ with a reduced SNC divisor D_S. Suppose that X is a nonsingular 3-fold, and $f : X \to S$ is a dominant morphism. X is 1-prepared (with respect to f) if $D_X = f^{-1}(D_S)_{red}$ is a SNC divisor on X which contains the locus where f is not smooth, and if C_1, C_2 are the two components of D_S whose intersection is nonempty, T_1 is a component of X dominating C_1 and T_2 is a component of D_X which dominates C_2, then T_1 and T_2 are disjoint.*

The following lemma is an easy consequence of the main theorem on resolution of singularities.

Lemma 2.2. *Suppose that $g : Y \to T$ is a dominant morphism of a 3-fold over $\mathfrak{k}$ to a surface over $\mathfrak{k}$ and D_T is a 1-cycle on T such that $g^{-1}(D_R)$ contains the locus where g is not smooth. Then there exists a commutative diagram of morphisms*

$$\begin{array}{ccc} Y_1 & \overset{g_1}{\rightarrow} & T_1 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ Y & \overset{g}{\rightarrow} & T \end{array}$$

such that the vertical arrows are products of blow ups of nonsingular subvarieties contained in the preimage of D_T, Y_1 and T_1 are nonsingular and $D_{T_1} = \pi_1^{-1}(D_T)$ is a SNC divisor on T_1 such that Y_1 is 1-prepared with respect to g_1.

For the duration of this paper, S will be a fixed nonsingular surface over $\mathfrak{k}$, with a (reduced) SNC divisor D_S. To simplify notation, we will often write D to denote D_X, if $f : X \to S$ is 1-prepared.

Suppose that X is 1-prepared with respect to $f : X \to S$. A permissible blow up of X is the blow up $\pi_1 : X_1 \to X$ of a point of D_X or a nonsingular curve contained in D_X which makes SNCs with D_X. Then $D_{X_1} = \pi_1^{-1}(D_X)_{red} = (f \circ \pi_1)^{-1}(X_S)_{red}$ is a SNC divisor on X_1 and X_1 is 1-prepared with respect to $f \circ \pi_1$.

Assume that X is 1-prepared with respect to D. We will say that $p \in X$ is a n-point (for D) if p is on exactly n components of D. Suppose $q \in D_S$ and u, v are regular parameters in $\mathcal{O}_{S,q}$ such that either $u = 0$ is a local equation of D_S at q or $uv = 0$ is a local equation of D_S at q. u, v are called permissible parameters at q.

For $p \in f^{-1}(q)$, we have regular parameters x, y, z in $\hat{\mathcal{O}}_{X,p}$ such that

1) If p is a 1-point,

$$u = x^a, v = P(x) + x^b F \tag{1}$$

where $x = 0$ is a local equation of D, $x \not| F$ and $x^b F$ has no terms which are a power of x.

2) If p is a 2-point, after possibly interchanging u and v,

$$u = (x^a y^b)^l, v = P(x^a y^b) + x^c y^d F \tag{2}$$

where $xy = 0$ is a local equation of D, $a, b > 0$, $\gcd(a, b) = 1$, $x, y \not| F$ and $x^c y^d F$ has no terms which are a power of $x^a y^b$.

3) If p is a 3-point, after possibly interchanging u and v,

$$u = (x^a y^b z^c)^l, v = P(x^a y^b z^c) + x^d y^e z^f F \tag{3}$$

where $xyz = 0$ is a local equation of D, $a, b, c > 0$, $\gcd(a, b, c) = 1$, $x, y, z \not| F$ and $x^d y^e z^f F$ has no terms which are a power of $x^a y^b z^c$.

regular parameters x, y, z in $\hat{\mathcal{O}}_{X,p}$ giving forms (1), (2) or (3) are called permissible parameters at p for u, v.

Suppose that X is 1-prepared. We define an ideal sheaf

$$\mathcal{I} = \text{ fitting ideal sheaf of the image of } f^* : \Omega^2_S \to \Omega^2_X(\log(D))$$

in $\mathcal{O}_X$. $\mathcal{I} = \mathcal{O}_X(-G)\overline{\mathcal{I}}$ where G is an effective divisor supported on D and $\overline{\mathcal{I}}$ has height ≥ 2.

Suppose that $E_1, \ldots, E_n$ are the irreducible components of D. For $p \in X$, define

$$\sigma_D(p) = \text{order}_{\mathcal{O}_{X,p}/(\sum_{p\in E_i} \mathcal{I}_{E_i,p})} \overline{\mathcal{I}_p} \left(\mathcal{O}_{X,p} / \sum_{p \in E_i} \mathcal{I}_{E_i,p} \right) \in \mathbb{N} \cup \{\infty\}.$$

Lemma 2.3. *σ_D is upper semicontinuous in the Zariski topology of the scheme X.*

Proof. For a fixed subset $J \subset \{1, 2, \ldots, n\}$, we have that the function

$$\text{order}_{\mathcal{O}_{X,p}/(\sum_{i\in J} \mathcal{I}_{E_i,p})} \overline{\mathcal{I}_p} \left(\mathcal{O}_{X,p} / \sum_{i \in J} \mathcal{I}_{E_i,p} \right)$$

is upper semicontinuous, and if $J \subset J' \subset \{1, 2, \ldots, n\}$. we have that

$$\text{order}_{\mathcal{O}_{X,p}/(\sum_{i\in J} \mathcal{I}_{E_i,p})} \overline{\mathcal{I}_p} \left(\mathcal{O}_{X,p} / \sum_{i \in J} \mathcal{I}_{E_i,p} \right) \leq \text{order}_{\mathcal{O}_{X,p}/(\sum_{i\in J'} \mathcal{I}_{E_i,p})} \overline{\mathcal{I}_p} \left(\mathcal{O}_{X,p} / \sum_{i \in J'} \mathcal{I}_{E_i,p} \right).$$

$\square$

Thus for $r \in \mathbb{N} \cup \{\infty\}$,

$$\text{Sing}_r(X) = \{p \in X \mid \sigma_D(p) \geq r\}$$

is a closed subset of X, which is supported on D and has dimension ≤ 1 if $r > 0$.

Definition 2.4. *A point $p \in X$ is prepared if $\sigma_D(p) = 0$.*

We have that $\sigma_D(p) = 0$ if and only if $\overline{\mathcal{I}}_p = \mathcal{O}_{X,p}$. Further,

$$\text{Sing}_1(X) = \{p \in X \mid \overline{\mathcal{I}}_p \neq \mathcal{O}_{X,p}\}.$$

If $p \in X$ is a 1-point with an expression (1) we have

$$(\overline{\mathcal{I}}_p + (x))\hat{\mathcal{O}}_{X,p} = (x, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}). \tag{4}$$

If $p \in X$ is a 2-point with an expression (2) we have

$$(\overline{\mathcal{I}}_p + (x,y))\hat{\mathcal{O}}_{X,p} = (x, y, (ad - bc)F, \frac{\partial F}{\partial z}). \tag{5}$$

If $p \in X$ is a 3-point with an expression (3) we have

$$(\overline{\mathcal{I}}_p + (x,y,z))\hat{\mathcal{O}}_{X,p} = (x, y, z, (ae - bd)F, (af - cd)F, (bf - ce)F). \tag{6}$$

If $p \in X$ is a 1-point with an expression (1), then $\sigma_D(p) = \text{ord } F(0,y,z) - 1$. We have $0 \le \sigma_D(p) < \infty$ if p is a 1-point. If $p \in X$ is a 2-point, we have

$$\sigma_D(p) = \begin{cases} 0 & \text{if ord } F(0,0,z) = 0 \text{ (in this case, } ad - bc \neq 0) \\ \text{ord } F(0,0,z) - 1 & \text{if } 1 \le \text{ord } F(0,0,z) < \infty \\ \infty & \text{if ord } F(0,0,z) = \infty. \end{cases}$$

If $p \in X$ is a 3-point, let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$

we have

$$\sigma_D(p) = \begin{cases} 0 & \text{if ord } F(0,0,0) = 0 \text{ (in this case, rank}(A) = 2) \\ \infty & \text{if ord } F(0,0,0) = \infty. \end{cases}$$

Lemma 2.5. *Suppose that X is 1-prepared and $\pi_1 : X_1 \to X$ is a toroidal morphism with respect to D. Then X_1 is 1-prepared and $\sigma_D(p_1) \le \sigma_D(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.*

Proof. Suppose that $p \in X$ is a 2-point and $p_1 \in \pi_1^{-1}(p)$. Then there exist permissible parameters x, y, z at p giving an expression (2). In $\hat{\mathcal{O}}_{X_1,p_1}$, there are regular parameters x_1, y_1, z where

$$x = x_1^{a_{11}}(y_1 + \alpha)^{a_{12}},\ y = x_1^{a_{21}}(y_1 + \alpha)^{a_{22}} \tag{7}$$

with $\alpha \in \mathfrak{k}$ and $a_{11}a_{22} - a_{12}a_{22} = \pm 1$. If $\alpha = 0$, so that p_1 is a 2-point, then x_1, y_1, z are permissible parameters at p_1 and substitution of (7) into (2) gives an expression of the form (2) at p_1, showing that $\sigma_D(p_1) \le \sigma_D(p)$. If $\alpha \neq 0 \in \mathfrak{k}$, so that p_1 is a 1-point, set $\lambda = \frac{aa_{12}+ba_{22}}{aa_{11}+ba_{21}}$ and $\overline{x}_1 = x_1(y_1 + \alpha)^{\lambda}$. Then $\overline{x}_1, y_1, z$ are permissible parameters at p_1. Substitution into (2) leads to a form (1) with $\sigma_D(p_1) \le \sigma_D(p)$.

If $p \in X$ is a 3-point and $\sigma_D(p) \neq \infty$, then $\sigma_D(p) = 0$ so that p is prepared. Thus there exist permissible parameters x, y, z at p giving an expression (3) with $F = 1$. Suppose that $p_1 \in \pi_1^{-1}(p)$. In $\hat{\mathcal{O}}_{X_1,p_1}$ there are regular parameters x_1, y_1, z_1 such that

$$\begin{array}{rcl} x & = & (x_1 + \alpha)^{a_{11}}(y_1 + \beta)^{a_{12}}(z_1 + \gamma)^{a_{13}} \\ y & = & (x_1 + \alpha)^{a_{21}}(y_1 + \beta)^{a_{22}}(z_1 + \gamma)^{a_{23}} \\ z & = & (x_1 + \alpha)^{a_{31}}(y_1 + \beta)^{a_{32}}(z_1 + \gamma)^{a_{33}} \end{array} \tag{8}$$

where at least one of $\alpha, \beta, \gamma \in \mathfrak{k}$ is zero. Substituting into (3), we find permissible parameters at p_1 giving a prepared form. $\square$

Suppose that X is 1-prepared with respect to $f : X \to S$. Define

$$\Gamma_D(X) = \max\{\sigma_D(p) \mid p \in X\}.$$

Lemma 2.6. *Suppose that X is 1-prepared and C is a 2-curve of D and there exists $p \in C$ such that $\sigma_D(p) < \infty$. Then $\sigma_D(q) = 0$ at the generic point q of C.*

Proof. If p is a 3-point then $\sigma_D(p)=0$ and the lemma follows from upper semicontinuity of σ_D.

Suppose that p is a 2-point. If $\sigma_D(p)=0$ then the lemma follows from upper semicontinuity of σ_D, so suppose that $0<\sigma_D(p)<\infty$. There exist permissible parameters x,y,z at p giving a form (2), such that x,y,z are uniformizing parameters on an étale cover U of an affine neighborhood of p. Thus for α in a Zariski open subset of $\mathfrak{k}$, $x,y,\overline{z}=z-\alpha$ are permissible parameters at a 2-point $\overline{p}$ of C. After possibly replacing U with a smaller neighborhood of p, we have

$$\frac{\partial F}{\partial z}=\frac{1}{x^cy^d}\frac{\partial v}{\partial z}\in\Gamma(U,\mathcal{O}_X)$$

and $\frac{\partial F}{\partial z}(0,0,z)\neq 0$. Thus there exists a 2-point $\overline{p}\in C$ with permissible parameters $x,y,\overline{z}=z-\alpha$ such that $\frac{\partial F}{\partial z}(0,0,\alpha)\neq 0$, and thus there is an expression (2) at $\overline{p}$

$$\begin{array}{rcl} u&=&(x^ay^b)^l\\ v&=&P_1(x^ay^b)+x^cy^dF_1(x,y,\overline{z})\end{array}$$

with ord $F_1(0,0,\overline{z})=0$ or 1, so that $\sigma_D(\overline{p})=0$. By upper semicontinuity of σ_D, $\sigma_D(q)=0$. □

Proposition 2.7. *Suppose that X is 1-prepared with respect to $f:X\rightarrow S$. Then there exists a toroidal morphism $\pi_1:X_1\rightarrow X$ with respect to D, such that π_1 is a sequence of blow ups of 2-curves and 3-points, and*

1) *$\sigma_D(p)<\infty$ for all $p\in D_{X_1}$.*
2) *X_1 is prepared (with respect to $f_1=f\circ\pi_1:X_1\rightarrow S$) at all 3-points and the generic point of all 2-curves of D_{X_1}.*

Proof. By upper semicontinuity of σ_D, Lemma 2.6 and Lemma 2.5, we must show that if $p\in X$ is a 3-point with $\sigma_D(p)=\infty$ then there exists a toroidal morphism $\pi_1:X_1\rightarrow X$ such that $\sigma_D(p_1)=0$ for all 3-points $p_1\in\pi_1^{-1}(p)$ and if $p\in X$ is a 2-point with $\sigma_D(p)=\infty$ then there exists a toroidal morphism $\pi_1:X_1\rightarrow X$ such that $\sigma_D(p_1)<\infty$ for all 2-points $p_1\in\pi_1^{-1}(p)$.

First suppose that p is a 3-point with $\sigma_D(p)=\infty$. Let x,y,z be permissible parameters at p giving a form (3). There exist regular parameters $\tilde{x},\tilde{y},\tilde{z}$ in $\mathcal{O}_{X,p}$ and unit series $\alpha,\beta,\gamma\in\hat{\mathcal{O}}_{X,p}$ such that $x=\alpha\tilde{x}$, $y=\beta\tilde{y}$, $z=\gamma\tilde{z}$. Write $F=\sum b_{ijk}x^iy^jz^k$ with $b_{ijk}\in\mathfrak{k}$. Let $I=(\tilde{x}^i\tilde{y}^j\tilde{z}^k\mid b_{ijk}\neq 0)$, an ideal in $\mathcal{O}_{X,p}$. Since $\tilde{x}\tilde{y}\tilde{z}=0$ is a local equation of D at p, there exists a toroidal morphism $\pi_1:X_1\rightarrow X$ with respect to D such that $I\mathcal{O}_{X_1,p_1}$ is principal for all $p_1\in\pi_1^{-1}(p)$. At a 3-point $p_1\in\pi_1^{-1}(p)$, there exist permissible parameters x_1,y_1,z_1 such that

$$\begin{array}{rcl} x&=&x_1^{a_{11}}y_1^{a_{12}}z_1^{a_{13}}\\ y&=&x_1^{a_{21}}y_1^{a_{22}}z_1^{a_{23}}\\ z&=&x_1^{a_{31}}y_1^{a_{32}}z_1^{a_{33}}\end{array}$$

with $\mathrm{Det}(a_{ij})=\pm 1$. Substituting into (3), we obtain an expression (3) at p_1, where

$$\begin{array}{rcl} u&=&(x_1^{a_1}y_1^{b_1}z_1^{c_1})^l\\ v&=&P_1(x_1^{a_1}y_1^{b_1}z_1^{c_1})+x_1^{d_1}y_1^{e_1}z_1^{f_1}F_1\end{array}$$

where $P_1(x_1^{a_1}y_1^{b_1}z_1^{c_1})=P(x^ay^bz^c)$ and

$$F(x,y,z)=x_1^{\overline{a}}y_1^{\overline{b}}z_1^{\overline{c}}F_1(x_1,y_1,z_1).$$

with $x_1^{\overline{a}}y_1^{\overline{b}}z_1^{\overline{c}}$ a generator of $I\hat{\mathcal{O}}_{X_1,p_1}$ and $F_1(0,0,0)\neq 0$. Thus $\sigma_D(p_1)=0$.

Now suppose that p is a 2-point and $\sigma_D(p)=\infty$. There exist permissible parameters x,y,z at p giving a form (2). Write $F=\sum a_i(x,y)z^i$, with $a_i(x,y)\in\mathfrak{k}[[x,y]]$ for all i. We necessarily have that no $a_i(x,y)$ is a unit series.

Let I be the ideal $I=(a_i(x,y)\mid i\in\mathbb{N})$ in $\mathfrak{k}[[x,y]]$. There exists a sequence of blow ups of 2-curves $\pi_1:X_1\to X$ such that $\hat{\mathcal{O}}_{X_1,p_1}$ is principal at all 2-points $p_1\in\pi_1^{-1}(p)$. There exist $x_1,y_1\in\mathcal{O}_{X_1,p_1}$ so that x_1,y_1,z are permissible parameters at p_1, and

$$x=x_1^{a_{11}}y_1^{a_{12}},\ y=x_1^{a_{21}}y_1^{a_{22}}$$

with $a_{11}a_{22}-a_{12}a_{21}=\pm1$. Let $x_1^{\overline{a}}y_1^{\overline{b}}$ be a generator of $I\mathcal{O}_{T_1,q_1}$. Then $F=x_1^{\overline{a}}y_1^{\overline{b}}F_1(x_1,y_1,z)$ where $F_1(0,0,z)\neq0$, and we have an expression (2) at p_1, where

$$\begin{array}{lll} u &=& (x_1^{a_1}y_1^{b_1})^{l_1}\\ v &=& P_1(x_1^{a_1}y_1^{b_1})+x_1^{d_1}y_1^{e_1}F_1 \end{array}$$

where $P_1(x_1^{a_1}y_1^{b_1})=P(x^ay^b)$. Thus $\sigma_D(p_1)<\infty$ and $\sigma_D(q)<\infty$ if q is the generic point of the 2-curve of D_{X_1} containing p_1.

$\square$

We will say that X is 2-prepared (with respect to $f:X\to S$) if it satisfies the conclusions of Proposition 2.7. We then have that $\Gamma_D(X)<\infty$.

If X is 2-prepared, we have that $\mathrm{Sing}_1(X)$ is a union of (closed) curves whose generic point is a 1-point and isolated 1-points and 2-points. Further, $\mathrm{Sing}_1(X)$ contains no 3-points.

3. 3-preparation

Lemma 3.1. *Suppose that X is 2-prepared. Suppose that $p\in X$ is such that $\sigma_D(p)>0$. Let $m=\sigma_D(p)+1$. Then there exist permissible parameters x,y,z at p such that there exist $\tilde{x},y\in\mathcal{O}_{X,p}$, an étale cover U of an affine neighborhood of p, such that $x,z\in\Gamma(U,\mathcal{O}_X)$ and x,y,z are uniformizing parameters on U, and $x=\gamma\tilde{x}$ for some unit series $\gamma\in\hat{\mathcal{O}}_{X,p}$. We have an expression (1) or (2), if p is respectively a 1-point or a 2-point, with*

$$F=\tau z^m+a_2(x,y)z^{m-2}+\cdots+a_{m-1}(x,y)z+a_m(x,y)\tag{9}$$

where $m\geq2$ and $\tau\in\hat{\mathcal{O}}_{X_1,p}=\mathfrak{k}[[x,y,z]]$ is a unit, and $a_i(x,y)\neq0$ for $i=m-1$ or $i=m$. Further, if p is a 1-point, then we can choose x,y,z so that $x=y=0$ is a local equation of a generic curve through p on D.

For all but finitely many points p in the set of 1-points of X, there is an expression (9) where

(10)

a_i is either zero or has an expression $a_i=\overline{a}_ix^{r_i}$ where $\overline{a}_i$ is a unit and $r_i>0$ for $2\leq i\leq m$, and $a_m=0$ or $a_m=x^{r_m}\overline{a}_m$ where $r_m>0$ and ord$(\overline{a}_m(0,y))=1$.

Proof. There exist regular parameters $\tilde{x},y,\overline{z}$ in $\mathcal{O}_{X,p}$ and a unit $\gamma\in\hat{\mathcal{O}}_{X,p}$ such that $x=\gamma\tilde{x},y,\overline{z}$ are permissible parameters at p, with $\mathrm{ord}(F(0,0,\overline{z}))=m$. Thus there exists an affine neighborhood $\mathrm{Spec}(A)$ of p such that $V=\mathrm{Spec}(R)$, where $R=A[\gamma^{\frac{1}{a}}]$ is an étale cover of $\mathrm{Spec}(A)$, $x,y,\overline{z}$ are uniformizing parameters on V, and $u,v\in\Gamma(V,\mathcal{O}_X)$. Differentiating with respect to the uniformizing parameters $x,y,\overline{z}$ in R, set

$$\tilde{z}=\frac{\partial^{m-1}F}{\partial\overline{z}^{m-1}}=\omega(\overline{z}-\varphi(x,y))\tag{11}$$

where $\omega \in \hat{\mathcal{O}}_{X,p}$ is a unit series, and $\varphi(x,y) \in \mathfrak{k}[[x,y]]$ is a nonunit series, by the formal implicit function theorem. Set $z = \overline{z} - \varphi(x,y)$. Since R is normal, after possibly replacing $\mathrm{Spec}(A)$ with a smaller affine neighborhood of p,

$$\tilde{z} = \frac{1}{x^b}\frac{\partial^{m-1}v}{\partial \overline{z}^{m-1}} \in R.$$

By Weierstrass preparation for Henselian local rings (Proposition 6.1 [37]), $\varphi(x,y)$ is integral over the local ring $\mathfrak{k}[x,y]_{(x,y)}$. Thus after possibly replacing A with a smaller affine neighborhood of p, there exists an étale cover U of V such that $\varphi(x,y) \in \Gamma(U, \mathcal{O}_X)$, and thus $z \in \Gamma(U, \mathcal{O}_X)$.

Let $G(x,y,z) = F(x,y,\overline{z})$. We have that

$$G = G(x,y,0) + \frac{\partial G}{\partial z}(x,y,0)z + \cdots + \frac{1}{(m-1)!}\frac{\partial^{m-1}G}{\partial z^{m-1}}(x,y,0)z^{m-1} + \frac{1}{m!}\frac{\partial^m G}{\partial z^m}(x,y,0)z^m + \cdots$$

We have

$$\frac{\partial^{m-1}G}{\partial z^{m-1}}(x,y,0) = \frac{\partial^{m-1}F}{\partial \overline{z}^{m-1}}(x,y,\varphi(x,y)) = 0$$

and

$$\frac{\partial^m G}{\partial z^m}(x,y,0) = \frac{\partial^m F}{\partial \overline{z}^m}(x,y,\varphi(x,y))$$

is a unit in $\hat{\mathcal{O}}_{X,p}$. Thus we have the desired form (9), but we must still show that $a_m \neq 0$ or $a_{m-1} \neq 0$. If $a_i(x,y) = 0$ for $i = m$ and $i = m-1$, we have that $z^2 \mid F$ in $\hat{\mathcal{O}}_{X,p}$, since $m \geq 2$. This implies that the ideal of 2×2 minors

$$I_2\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} \subset (z),$$

which implies that $z = 0$ is a component of D which is impossible. Thus either $a_{m-1} \neq 0$ or $a_m \neq 0$.

Suppose that C is a curve in $\mathrm{Sing}_1(X)$ (containing a 1-point) and $p \in C$ is a general point. Let $r = \sigma_D(p)$. Set $m = r+1$. Let $x, y, \overline{z}$ be permissible parameters at p with $y, \overline{z} \in \mathcal{O}_{X,p}$, which are uniformizing parameters on an étale cover U of an affine neighborhood of p such that $x = \overline{z} = 0$ are local equations of C and we have a form (1) at p with

$$F = \tau\overline{z}^m + a_1(x,y)\overline{z}^{m-1} + \cdots + a_m(x,y). \tag{12}$$

For α in a Zariski open subset of $\mathfrak{k}$, $x, \overline{y} = y - \alpha, \overline{z}$ are permissible parameters at a point $q \in C \cap U$. For most points q on the curve $C \cap U$, we have that $a_i(x,y) = x^{r_i}\overline{a}_i(x,y)$ where $\overline{a}_i(x,y)$ is a unit or zero for $1 \leq i \leq m-1$ in $\hat{\mathcal{O}}_{X,q}$. Since $\sigma_D(p) = r$ at this point, we have that $1 \leq r_i$ for all i. We further have that if $a_m \neq 0$, then $a_m = x^{r_m}a'$ where $a' = f(y) + x\Omega$ where $f(y)$ is non constant. Thus

$$0 \neq \frac{\partial a_m}{\partial y}(0,y) = \frac{\partial F}{\partial y}(0,y,0).$$

After possibly replacing U with a smaller neighborhood of p, we have

$$\frac{\partial F}{\partial y} = \frac{1}{x^b}\frac{\partial v}{\partial y} \in \Gamma(U, \mathcal{O}_X).$$

Thus $\frac{\partial a_m}{\partial y}(0,\alpha)\neq 0$ for most $\alpha\in\mathfrak{k}$. Since $r>0$, we have that $r_m>0$, and thus $r_i>0$ for all i in (12). We have

$$\frac{\partial^{m-1}F}{\partial\overline{z}^{m-1}}=\xi\overline{z}+a_1(x,y),$$

where ξ is a unit series. Comparing the above equation with (11), we observe that $\varphi(x,y)$ is a unit series in x and y times $a_1(x,y)$. Thus x divides $\varphi(x,y)$. Setting $z=\overline{z}-\varphi(x,y)$, we obtain an expression (9) such that x divides a_i for all i. Now argue as in the analysis of (12), after substituting $z=\overline{z}-\varphi(x,y)$, to conclude that there is an expression (9), where (10) holds at most points $q\in C\cap U$. Thus a form (9) and (10) holds at all but finitely many 1-points of X.

□

Lemma 3.2. *Suppose that X is 2-prepared, C is a curve in $Sing_1(X)$ containing a 1-point and p is a general point of C. Let $m=\sigma_D(p)+1$. Suppose that $\tilde{x},y\in\mathcal{O}_{X,p}$ are such that $\tilde{x}=0$ is a local equation of D at p and the germ $\tilde{x}=y=0$ intersects C transversally at p. Then there exists an étale cover U of an affine neighborhood of p and $z\in\Gamma(U,\mathcal{O}_X)$ such that $\tilde{x},y,z$ give a form (9) at p.*

Proof. There exists $\overline{z}\in\mathcal{O}_{X,p}$ such that $\tilde{x},y,\overline{z}$ are regular parameters in $\mathcal{O}_{X,p}$ and $x=\overline{z}=0$ is a local equation of C at p. There exists a unit $\gamma\in\hat{\mathcal{O}}_{X,p}$ such that $x=\gamma\tilde{x},y,\overline{z}$ are permissible parameters at p. We have an expression of the form (1),

$$u=x^a,v=P(x)+x^bF$$

at p. Write $F=f(y,\overline{z})+x\Omega$ in $\hat{\mathcal{O}}_{X,p}$. Let I be the ideal in $\hat{\mathcal{O}}_{X,p}$ generated by x and

$$\{\frac{\partial^{i+j}f}{\partial y^i\partial\overline{z}^j}\mid 1\le i+j\le m-1\}.$$

The radical of I is the ideal $(x,\overline{z})$, as $x=\overline{z}=0$ is a local equation of $\mathrm{Sing}_{m-1}(X)$ at p. Thus $\overline{z}$ divides $\frac{\partial^{i+j}}{\partial y^i\partial\overline{z}^j}$ for $1\le i+j\le m-1$ (with $m\ge 2$). Expanding

$$f=\sum_{i=0}^{\infty}b_i(y)\overline{z}^i$$

(where $b_0(0)=0$) we see that $\frac{\partial b_0}{\partial y}=0$ (so that $b_0(y)=0$) and $b_i(y)=0$ for $1\le i\le m-1$. Thus $\overline{z}^m$ divides $f(y,\overline{z})$. Since $\sigma_D(p)=m-1$, we have that $f=\tau\overline{z}^m$ where τ is a unit series. Thus $x,y,\overline{z}$ gives a form (1) with $\mathrm{ord}(F(0,0,\overline{z}))=m$. Now the proof of Lemma 3.1 gives the desired conclusion. □

Let $\omega(m,r_2,\ldots,r_{m-1})$ be a function which associates a positive integer to a positive integer m, natural numbers $r_2,\ldots,r_{m-2}$ and a positive integer r_{m-1}. We will give a precise form of ω after Theorem 4.1.

Definition 3.3. *X is 3-prepared (with respect to $f:X\to S$) at a point $p\in D$ if $\sigma_D(p)=0$ or if $\sigma_D(p)>0$, f is 2-prepared with respect to D at p and there are permissible parameters x,y,z at p such that x,y,z are uniformizing parameters on an étale cover of an affine neighborhood of p and we have one of the following forms, with $m=\sigma_D(p)+1$:*

1) *p is a 2-point, and we have an expression (2) with*

(13) $$F=\tau_0z^m+\tau_2x^{r_2}y^{s_2}z^{m-2}+\cdots+\tau_{m-1}x^{r_{m-1}}y^{s_{m-1}}z+\tau_mx^{r_m}y^{s_m}$$

where $\tau_0\in\hat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i\in\hat{\mathcal{O}}_{X,p}$ are units (or zero), $r_i+s_i>0$ whenever $\tau_i\neq 0$ and $(r_m+c)b-(s_m+d)a\neq 0$. Further, $\tau_{m-1}\neq 0$ or $\tau_m\neq 0$.

2) *p is a 1-point, and we have an expression (1) with*

$$F = \tau_0 z^m + \tau_2 x^{r_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} z + \tau_m x^{r_m} \tag{14}$$

where $\tau_0 \in \hat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \hat{\mathcal{O}}_{X,p}$ are units (or zero) for $2 \le i \le m-1$, $\tau_m \in \hat{\mathcal{O}}_{X,p}$ and $ord(\tau_m(0,y,0)) = 1$ (or $\tau_m = 0$). Further, $r_i > 0$ if $\tau_i \ne 0$, and $\tau_{m-1} \ne 0$ or $\tau_m \ne 0$.

3) *p is a 1-point, and we have an expression (1) with*

$$F = \tau_0 z^m + \tau_2 x^{r_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} z + x^t \Omega \tag{15}$$

where $\tau_0 \in \hat{\mathcal{O}}_{X,p}$ is a unit, $\tau_i \in \hat{\mathcal{O}}_{X,p}$ are units (or zero) for $2 \le i \le m-1$, $\Omega \in \hat{\mathcal{O}}_{X,p}$, $\tau_{m-1} \ne 0$ and $t > \omega(m, r_2, \ldots, r_{m-1})$ (where we set $r_i = 0$ if $\tau_i = 0$). Further, $r_i > 0$ if $\tau_i \ne 0$.

X is 3-prepared if X is 3-prepared for all $p \in X$.

Lemma 3.4. *Suppose that X is 2-prepared with respect to $f : X \to S$. Then there exists a sequence of blow ups of 2-curves $\pi_1 : X \to X_1$ such that X_1 is 3-prepared with respect to $f \circ \pi_1$, except possibly at a finite number of 1-points.*

Proof. The conclusions follow from Lemmas 3.1, 2.6 and 2.5, and the method of analysis above 2-points of the proof of 2.7. □

Lemma 3.5. *Suppose that $u, v \in \mathfrak{k}[[x,y]]$. Let $T_0 = Spec(\mathfrak{k}[[x,y]])$. Suppose that $u = x^a$ for some $a \in \mathbb{Z}_+$, or $u = (x^a y^b)^l$ where $gcd(a,b) = 1$ for some $a, b, l \in \mathbb{Z}_+$. Let $p \in T_0$ be the maximal ideal (x,y). Suppose that $v \in (x,y)\mathfrak{k}[[x,y]]$. Then either $v \in \mathfrak{k}[[x]]$ or there exists a sequence of blow ups of points $\lambda : T_1 \to T_0$ such that for all $p_1 \in \lambda^{-1}(p)$, we have regular parameters x_1, y_1 in $\hat{\mathcal{O}}_{T_1,p_1}$, regular parameters $\tilde{x}_1, \tilde{y}_1$ in $\mathcal{O}_{T_1,p_1}$ and a unit $\gamma_1 \in \hat{\mathcal{O}}_{T_1,p_1}$ such that $x_1 = \gamma_1 \tilde{x}_1$, and one of the following holds:*

1)
$$u = x_1^{a_1}, v = P(x_1) + x_1^b y_1^c$$

with $c > 0$ or

2) *There exists a unit $\gamma_2 \in \hat{\mathcal{O}}_{T_1,p_1}$ such that $y_1 = \gamma_2 \tilde{y}_1$ and*

$$u = (x_1^{a_1} y_1^{b_1})^{\ell_1}, v = P(x_1^{a_1} y_1^{b_1}) + x_1^{c_1} y_1^{d_1}$$

with $gcd(a_1, b_1) = 1$ and $a_1 d_1 - b_1 c_1 \ne 0$.

Proof. Let

$$J = \mathrm{Det} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

First suppose that $J = 0$. Expand $v = \sum \gamma_{ij} x^i y^j$ with $\gamma_{ij} \in \mathfrak{k}$. If $u = x^a$, then $\sum j \gamma_{ij} x^i y^{j-1} = 0$ implies $\gamma_{ij} = 0$ if $j > 0$. Thus $v = P(x) \in \mathfrak{k}[[x]]$. If $u = (x^a y^b)^l$, then

$$0 = J = l x^{la-1} y^{lb-1} (\sum_{i,j} (ja - ib) \gamma_{ij} x^i y^j)$$

implies $\gamma_{ij} = 0$ if $ja - ib \ne 0$, which implies that $v \in \mathfrak{k}[[x^a y^b]]$.

Now suppose that $J \ne 0$. Let E be the divisor $uJ = 0$ on T_0. There exists a sequence of blow ups of points $\lambda : T_1 \to T_0$ such that $\lambda^{-1}(E)$ is a SNC divisor on T_1. Suppose that

$p_1 \in \lambda^{-1}(p)$. There exist regular parameters $\tilde{x}_1, \tilde{y}_1$ in $\hat{\mathcal{O}}_{T_1,p_1}$ such that if

$$J_1 = \mathrm{Det}\begin{pmatrix} \frac{\partial u}{\partial \tilde{x}_1} & \frac{\partial u}{\partial \tilde{y}_1} \\ \frac{\partial v}{\partial \tilde{x}_1} & \frac{\partial v}{\partial \tilde{y}_1} \end{pmatrix},$$

then

$$u = \tilde{x}_1^{a_1},\ J_1 = \delta \tilde{x}_1^{b_1}\tilde{y}_1^{c_1} \tag{16}$$

where $a_1 > 0$ and δ is a unit in $\hat{\mathcal{O}}_{T_1,p_1}$, or

$$u = (\tilde{x}_1^{a_1}\tilde{y}_1^{b_1})^{l_1},\ J_1 = \delta \tilde{x}_1^{c_1}\tilde{y}_1^{d_1} \tag{17}$$

where $a_1, b_1 > 0$, $\gcd(a_1, b_1) = 1$ and δ is a unit in $\hat{\mathcal{O}}_{T_1,p_1}$. Expand $v = \sum \gamma_{ij}\tilde{x}_1^i\tilde{y}_1^j$ with $\gamma_{ij} \in \mathfrak{k}$.

First suppose (16) holds. Then

$$a_1 x_1^{a_1-1}\left(\sum_{i,j} j\gamma_{ij}\tilde{x}_1^i\tilde{y}_1^{j-1}\right) = \delta\tilde{x}_1^{b_1}\tilde{y}_1^{c_1}.$$

Thus $v = P(\tilde{x}_1) + \varepsilon\tilde{x}_1^e\tilde{y}_1^f$ where $P(\tilde{x}_1) \in \mathfrak{k}[[\tilde{x}_1]]$, $e = b_1 - a_1 + a$, $f = c_1 + 1$ and ε is a unit series. Since $f > 0$, we can make a formal change of variables, multiplying $\tilde{x}_1$ by an appropriate unit series to get the form 1) of the conclusions of the lemma.

Now suppose that (17) holds. Then

$$\tilde{x}_1^{a_1l_1-1}\tilde{y}_1^{b_1l_1-1}\left(\sum_{ij}(a_1l_1j - b_1l_1i)\gamma_{ij}\tilde{x}_1^i\tilde{y}_1^j\right) = \delta\tilde{x}_1^{c_1}\tilde{y}_1^{d_1}.$$

Thus $v = P(\tilde{x}_1^{a_1}\tilde{y}_1^{b_1}) + \varepsilon\tilde{x}_1^e\tilde{y}_1^f$, where P is a series in $\tilde{x}_1^{a_1}\tilde{y}_1^{b_1}$, ε is a unit series, $e = c_1+1-a_1l_1$, $f = d_1 + 1 - b_1l_1$. Since $a_1l_1f - b_1l_1e \neq 0$, we can make a formal change of variables to reach 2) of the conclusions of the lemma. □

Lemma 3.6. *Suppose that X is 2-prepared with respect to $f : X \to S$. Suppose that $p \in D$ is a 1-point with $m = \sigma_D(p) + 1 > 1$. Let u, v be permissible parameters for $f(p)$ and x, y, z be permissible parameters for D at p such that a form (9) holds at p. Let U be an étale cover of an affine neighborhood of p such that x, y, z are uniformizing parameters on U. Let C be the curve in U which has local equations $x = y = 0$ at p.*

Let $T_0 = Spec(\mathfrak{k}[x, y])$, $\Lambda_0 : U \to T_0$. Then there exists a sequence of quadratic transforms $T_1 \to T_0$ such that if $U_1 = U \times_{T_0} T_1$ and $\psi_1 : U_1 \to U$ is the induced sequence of blow ups of sections over C, $\Lambda_1 : U_1 \to T_1$ is the projection, then U_1 is 2-prepared with respect to $f \circ \psi_1$ at all $p_1 \in \psi_1^{-1}(p)$. Further, for every point $p_1 \in \psi_1^{-1}(p)$, there exist regular parameters x_1, y_1 in $\hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ such that x_1, y_1, z are permissible parameters at p_1, and there exist regular parameters $\tilde{x}_1, \tilde{y}_1$ in $\mathcal{O}_{T_1,\Lambda_1(p_1)}$ such that if p_1 is a 1-point, $x_1 = \alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1$ where $\alpha(\tilde{x}_1, \tilde{y}_1) \in \hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ is a unit series and $y_1 = \beta(\tilde{x}_1, \tilde{y}_1)$ with $\beta(\tilde{x}_1, \tilde{y}_1) \in \hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$, and if p_1 is a 2-point, then $x_1 = \alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1$ and $y_1 = \beta(\tilde{x}_1, \tilde{y}_1)\tilde{y}_1$, where $\alpha(\tilde{x}_1, \tilde{y}_1), \beta(\tilde{x}_1, \tilde{y}_1) \in \hat{\mathcal{O}}_{T_1,\Lambda_1(p_1)}$ are unit series. We have one of the following forms:

1) *p_1 is a 2-point, and we have an expression (2) with*

$$F = \tau z^m + \overline{a}_2(x_1, y_1)x_1^{r_2}y_1^{s_2}z^{m-2} + \cdots + \overline{a}_{m-1}(x_1, y_1)x_1^{r_{m-1}}y_1^{s_{m-1}}z + \overline{a}_m x_1^{r_m}y_1^{s_m} \tag{18}$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\overline{a}_i(x_1,y_1) \in \mathfrak{k}[[x_1,y_1]]$ are units (or zero) for $2 \le i \le m-1$, $\overline{a}_m = 0$ or 1 and if $\overline{a}_m = 0$, then $\overline{a}_{m-1} \ne 0$. Further, $r_i + s_i > 0$ whenever $\overline{a}_i \ne 0$ and $a(r_m+c)b - (s_m+d)a \ne 0$.

2) *p_1 is a 1-point, and we have an expression (1) with*

(19) $$F = \tau z^m + \overline{a}_2(x_1,y_1)x_1^{r_2}z^{m-2} + \cdots + \overline{a}_{m-1}(x_1,y_1)x_1^{r_{m-1}}z + x_1^{r_m}y_1$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\overline{a}_i(x_1,y_1) \in \mathfrak{k}[[x_1,y_1]]$ are units (or zero) for $2 \le i \le m-1$. Further, $r_i > 0$ (whenever $\overline{a}_i \ne 0$).

3) *p_1 is a 1-point, and we have an expression (1) with*

(20) $$F = \tau z^m + \overline{a}_2(x_1,y_1)x_1^{r_2}z^{m-2} + \cdots + \overline{a}_{m-1}(x_1,y_1)x_1^{r_{m-1}}z + x_1^t y_1 \Omega$$

where $\tau \in \hat{\mathcal{O}}_{U_1,p_1}$ is a unit, $\overline{a}_i(x_1,y_1) \in \mathfrak{k}[[x_1,y_1]]$ are units (or zero) for $2 \le i \le m-1$ and $r_i > 0$ whenever $\overline{a}_i \ne 0$. We also have $t > \omega(m, r_2, \ldots, r_{m-1})$. Further, $\overline{a}_{m-1} \ne 0$ and $\Omega \in \hat{\mathcal{O}}_{U_1,p_1}$.

Proof. Let $\overline{p} = \Lambda_0(p)$. Let $T = \{i \mid a_i(x,y) \ne 0 \text{ and } 2 \le i < m\}$. There exists a sequence of blow ups $\varphi_1 : T_1 \to T_0$ of points over $\overline{p}$ such that at all points $q \in \psi_1^{-1}(p)$, we have permissible parameters x_1, y_1, z such that x_1, y_1 are regular parameters in $\hat{\mathcal{O}}_{T_1,\Lambda_1(q)}$ and we have that u is a monomial in x_1 and y_1 times a unit in $\hat{\mathcal{O}}_{T_1,\Lambda_1(q)}$, where $g = \prod_{i\in T} a_i(x,y)$.

Suppose that $a_m(x,y) \ne 0$. Let $\overline{v} = x^b a_m(x,y)$ if (1) holds and $\overline{v} = x^c y^d a_m(x,y)$ if (2) holds. We have $\overline{v} \notin \mathfrak{k}[[x]]$ (respectively $\overline{v} \notin \mathfrak{k}[[x^a y^b]]$). Then by Theorem 3.5 applied to $u, \overline{v}$, we have that there exists a further sequence of blow ups $\varphi_2 : T_2 \to T_1$ of points over $\overline{p}$ such that at all points $q \in (\psi_1 \circ \psi_2)^{-1}(p)$, we have permissible parameters x_2, y_2, z such that x_2, y_2 are regular parameters in $\hat{\mathcal{O}}_{T_2,\Lambda_2(q)}$ such that $u = 0$ is a SNC divisor and either

$$u = x_2^{\overline{a}}, \overline{v} = \overline{P}(x_2) + x_2^{\overline{b}}\overline{y}_2^{\overline{c}}$$

with $\overline{c} > 0$ or

$$u = (x_2^{\overline{a}}y_2^{\overline{b}})^t, \overline{v} = \overline{P}(x_2^{\overline{a}}y_2^{\overline{b}}) + x_2^{\overline{c}}\overline{y}_2^{\overline{d}}$$

where $\overline{a}\overline{d} - \overline{b}\overline{c} \ne 0$.

If q is a 2-point, we have thus achieved the conclusions of the lemma. Further, there are only finitely many 1-points q above p on U_2 where the conclusions of the lemma do not hold. At such a 1-point q, F has an expression

(21) $$F = \tau z^m + \overline{a}_2(x_2,y_2)x_2^{r_2}y_2^{s_2}z^{m-2} + \cdots + \overline{a}_{m-1}(x_2,y_2)x_2^{r_{m-1}}y_2^{s_{m-1}}z + \overline{a}_m x_2^{r_m}y_2^{s_m}$$

where $\overline{a}_m = 0$ or 1, $\overline{a}_i$ are units (or zero) for $2 \le i \le m$.

Let

$$J = I_2\begin{pmatrix} \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial y_2} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial y_2} & \frac{\partial v}{\partial z} \end{pmatrix} = x^n\left(\frac{\partial F}{\partial y_2}, \frac{\partial F}{\partial z}\right)$$

for some positive integer n. Since D contains the locus where f is not smooth, we have that the localization $J_{\mathfrak{p}} = (\hat{\mathcal{O}}_{U_2,q})_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal (y_2, z_2) in $\hat{\mathcal{O}}_{U_2,q}$.

We compute

$$\frac{\partial F}{\partial z} = \overline{a}_{m-1}x_2^{r_{m-1}}y_2^{s_{m-1}} + \Lambda_1 z$$

and

$$\frac{\partial F}{\partial y_2} = s_m\overline{a}_m y_2^{s_m-1}x_2^{r_m} + \Lambda_2 z$$

for some $\Lambda_1, \Lambda_2 \in \hat{\mathcal{O}}_{U_2,q}$, to see that either $\overline{a}_{m-1} \ne 0$ and $s_{m-1} = 0$, or $\overline{a}_m \ne 0$ and $s_m = 1$.

Let q be one of these points, and let $\varphi_3 : T_3 \rightarrow T_2$ be the blow up of $\Lambda_2(q)$. We then have that the conclusions of the lemma hold in the form (18) at the 2-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3y_3$ and $y_2 = y_3$. At a 1-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3, y_2 = x_3(y_3 + \alpha)$ with $\alpha \neq 0$, we have that a form (19) holds. Thus the only case where we may possibly have not achieved the conclusions of the lemma is at the 1-point which has permissible parameters x_3, y_3, z defined by $x_2 = x_3$ and $y_2 = x_3y_3$. We continue to blow up, so that there is at most one point where the conclusions of the lemma do not hold. This point is a 1-point, which has permissible parameters x_3, y_3, z where $x_2 = x_3$ and $y_2 = x_3^n y_3$ where we can take n as large as we like. We thus have a form

$$u = x_3^a, v = P(x_3) + x_3^b F_3 \tag{22}$$

with $F_3 = \tau z^m + \overline{b}_2 x_3^{r_2} z^{m-2} + \cdots + \overline{b}_{m-1} x_3^{r_{m-2}} z + x_3^t \Omega$, where either $\overline{b}_i(x_3, y_3)$ is a unit or is zero, $\overline{b}_{m-1} \neq 0$, and $t > \omega(m, r_2, \ldots, r_{m-1})$ if $\overline{a}_{m-1} \neq 0$ and $s_{m-1} = 0$ which is of the form of (20), or we have a form (19) (after replacing y_3 with y_3 times a unit series in x_3 and y_3) if $\overline{a}_m \neq 0$ and $s_m = 1$.

$\square$

Lemma 3.7. *Suppose that X is 2-prepared with respect to $f : X \rightarrow S$. Suppose that $p \in D$ is a 1-point with $\sigma_D(p) > 0$. Let $m = \sigma_D(p) + 1$. Let x, y, z be permissible parameters for D at p such that a form (9) holds at p.*

Let notation be as in Lemma 3.6. For $p_1 \in \psi_1^{-1}(p)$ let $\overline{r}(p_1) = m + 1 + r_m$, if a form (19) holds at p_1, and

$$\overline{r}(p_1) = \begin{cases} max\{m + 1 + r_m, m + 1 + s_m\} & \text{if } \overline{a}_m = 1 \\ max\{m + 1 + r_{m-1}, m + 1 + s_{m-1}\} & \text{if } \overline{a}_m = 0 \end{cases}$$

if a form (18)holds at p_1. Let $\overline{r}(p_1) = m + 1 + r_{m-1}$ if a form (20) holds at p_1.

Let $r' = max\{\overline{r}(p_1) \mid p_1 \in \psi_1^{-1}(p)\}$. Let

$$r = r(p) = m + 1 + r'. \tag{23}$$

Suppose that $x^* \in \mathcal{O}_{X,p}$ is such that $x = \overline{\gamma} x^*$ for some unit $\overline{\gamma} \in \hat{\mathcal{O}}_{X,p}$ with $\overline{\gamma} \equiv 1 \mod m_p^r \hat{\mathcal{O}}_{X,p}$.

Let V be an affine neighborhood of p such that $x^*, y \in \Gamma(V, \mathcal{O}_X)$, and let C^* be the curve in V which has local equations $x^* = y = 0$ at p.

Let $T_0^* = Spec(\mathfrak{k}[x^*, y])$. Then there exists a sequence of blow ups of points $T_1^* \rightarrow T_0^*$ above (x^*, y) such that if $V_1 = V \times_{T_0^*} T_1^*$ and $\psi_1^* : V_1 \rightarrow V$ is the induced sequence of blow ups of sections over C^*, $\Lambda_1^* : V_1 \rightarrow T_1^*$ is the projection, then V_1 is 2-prepared at all $p_1^* \in (\psi_1^*)^{-1}(p)$. Further, for every point $p_1^* \in (\psi_1^*)^{-1}(p)$, there exist $\hat{x}_1, \overline{y}_1 \in \hat{\mathcal{O}}_{V_1, p_1^*}$ such that $\hat{x}_1, \overline{y}_1, z$ are permissible parameters at p_1^* and we have one of the following forms:

1) *p_1^* is a 2-point, and we have an expression (2) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} \overline{y}_1^{s_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \hat{x}_1^{r_{m-1}} \overline{y}_1^{s_{m-1}} z + \overline{\tau}_m \hat{x}_1^{r_m} \overline{y}_1^{s_m} \tag{24}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1, p_1^*}$ is a unit, $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1, p_1^*}$ are units (or zero) for $0 \leq i \leq m-1$, $\overline{\tau}_m$ is zero or 1, $\overline{\tau}_{m-1} \neq 0$ if $\overline{\tau}_m = 0$, $r_i + s_i > 0$ if $\overline{\tau}_i \neq 0$, and

$$(r_m + c)b - (s_m + d)a \neq 0.$$

2) *p_1^* is a 1-point, and we have an expression (1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} z^{m-2} + \cdots + \overline{\tau}_{m-1} \hat{x}_1^{r_{m-1}} z + \overline{\tau}_m \hat{x}_1^{r_m} \tag{25}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ is a unit, $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero), and ord$(\overline{\tau}_m(0,\overline{y}_1,0) = 1$. Further, $r_i > 0$ if $\overline{\tau}_i \neq 0$.

3) *p_1^* is a 1-point, and we have an expression (1) with*

$$F = \overline{\tau}_0 z^m + \overline{\tau}_2 \hat{x}_1^{r_2} z^{m-2} + \cdots + \overline{\tau}_{m-1}\hat{x}_1^{r_{m-1}} z + x_1^t \overline{\Omega} \tag{26}$$

where $\overline{\tau}_0 \in \hat{\mathcal{O}}_{V_1,p_1^*}$ is a unit, $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero), $\overline{\Omega} \in \hat{\mathcal{O}}_{V_1,p_1^*}$, $\overline{\tau}_{m-1} \neq 0$ and $t > \omega(m, r_2, \ldots, r_{m-1})$. Further, $r_i > 0$ if $\overline{\tau}_i \neq 0$.

Proof. The isomorphism $T_0^* \to T_0$ obtained by substitution of x^* for x and subsequent base change by the morphism $T_1 \to T_0$ of Lemma 3.6, induces a sequence of blow ups of points $T_1^* \to T_0^*$. The base change $\psi_1^* : V_1 = V \times_{T_0^*} T_1^* \to V \cong V \times_{T_0^*} T_0^*$ factors as a sequence of blow ups of sections over C^*. Let $\Lambda_1^* : V_1 \to T_1^*$ be the natural projection.

Let $p_1^* \in (\psi_1^*)^{-1}(p)$, and let $p_1 \in \psi_1^{-1}(p) \subset U_1$ be the corresponding point.

First suppose that p_1 has a form (19). With the notation of Lemma 3.6, we have polynomials φ, ψ such that

$$x = \varphi(\tilde{x}_1, \tilde{y}_1), y = \psi(\tilde{x}_1, \tilde{y}_1)$$

determines the birational extension $\mathcal{O}_{T_0,p_0} \to \mathcal{O}_{T_1,\Lambda_1(p_1)}$, and we have a formal change of variables

$$x_1 = \alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1, y_1 = \beta(\tilde{x}_1, \tilde{y}_1)$$

for some unit series α and series β. We further have expansions

$$a_i(x, y) = x_1^{r_i}\overline{a}_i(x_1, y_1)$$

for $2 \leq i \leq m-1$ where $\overline{a}_i(x_1, y_1)$ are unit series or zero, and

$$a_m(x, y) = x_1^{r_m} y_1.$$

We have $x = \overline{\gamma}x^*$ with $\overline{\gamma} \equiv 1 \bmod m_p^r\hat{\mathcal{O}}_{X,p}$. Set $y^* = y$. At p_1^*, we have regular parameters x_1^*, y_1^* in $\mathcal{O}_{T_1^*,\Lambda_1^*(p_1^*)}$ such that

$$x^* = \varphi(x_1^*, y_1^*), y^* = \psi(x_1^*, y_1^*),$$

and $x_1^*, y_1^*, \tilde{z}$ are regular parameters in $\mathcal{O}_{V_1,\overline{p}_1^*}$ (recall that $z = \sigma\tilde{z}$ in Lemma 3.1). We have regular parameters $\overline{x}_1, \overline{y}_1, \in \hat{\mathcal{O}}_{T_1^*,\Lambda_1^*(p_1^*)}$ defined by

$$\overline{x}_1 = \alpha(x_1^*, y_1^*)x_1^*, \overline{y}_1 = \beta(x_1^*, y_1^*).$$

We have $u = x^a = x_1^{a_1}$ where $a_1 = ad$ for some $d \in \mathbb{Z}_+$. Since $[\alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1]^d = x$, we have that $[\alpha(x_1^*, y_1^*)x_1^*]^d = x^*$. Set $\hat{x}_1 = \overline{\gamma}^{\frac{1}{d}}\overline{x}_1 = \overline{\gamma}^{\frac{1}{d}}\alpha(x_1^*, y_1^*)x_1^*$. We have that $\overline{\gamma}^{\frac{1}{d}}\alpha(x_1^*, y_1^*)$ is a unit in $\hat{\mathcal{O}}_{V_1,p_1^*}$, and $x = \hat{x}_1^d$. Thus $x_1 = \hat{x}_1$ (with an appropriate choice of root $\overline{\gamma}^{\frac{1}{d}}$). We have $u = \hat{x}_1^{ad}$, so that $\hat{x}_1, \overline{y}_1, z$ are permissible parameters at p_1^*.

For $2 \leq i \leq m-1$, we have

$$a_i(x, y) = a_i(\overline{\gamma}x^*, y^*) \equiv a_i(x^*, y^*) \bmod m_p^r\hat{\mathcal{O}}_{V,p}$$

and

$$\begin{array}{rcl} a_i(x^*, y^*) & = & a_i(\varphi(x_1^*, y_1^*), \psi(x_1^*, y_1^*)) \\ & = & \overline{x}_1^{r_i}\overline{a}_i(\overline{x}_1, \overline{y}_1) \\ & \equiv & x_1^{r_i}\overline{a}_i(x_1, \overline{y}_1) \bmod m_p^r\mathcal{O}_{V_1,p_1^*}. \end{array}$$

We further have

$$a_m(x^*, y^*) \equiv x_1^{r_m}\overline{y}_1 \bmod m_p^r\hat{\mathcal{O}}_{V_1,p_1^*}.$$

Thus we have expressions

$$(27)\qquad \begin{array}{rcl} u &=& x_1^{da}\\ v &=& P(x_1^d)+x_1^{bd}P_1(x_1)+x_1^{bd}(\overline{\tau}z^m+x_1^{r_2}\overline{a}_2(x_1,\overline{y}_1)z^{m-2}+\cdots+x_1^{r_m}\overline{y}_1+h)\end{array}$$

where $\overline{\tau}\in\hat{\mathcal{O}}_{V_1,p_1^*}$ is a unit series and

$$h\in m_p^r\hat{\mathcal{O}}_{V_1,p_1^*}\subset(x_1,z)^r.$$

Set $s=r-m$, and write

$$\begin{array}{rcl} h &=& z^m\Lambda_0(x_1,\overline{y}_1,z)+z^{m-1}x_1^{1+s}\Lambda_1(x_1,\overline{y}_1)+z^{m-2}x_1^{2+s}\Lambda_2(x_1,\overline{y}_1)+\cdots\\ && +zx_1^{(m-1)+s}\Lambda_{m-1}(x_1,\overline{y}_1)+x_1^{m+s}\Lambda_m(x_1,\overline{y}_1)\end{array}$$

with $\Lambda_0\in m_{p_1^*}\hat{\mathcal{O}}_{V_1,p_1^*}$ and $\Lambda_i\in\mathfrak{k}[[x_1,\overline{y}_1]]$ for $1\le i\le m$.

Substituting into (27), we obtain an expression

$$\begin{array}{rcl} u &=& x_1^{da}\\ v &=& P(x_1^d)+x_1^{bd}P_1(x_1)+x_1^{bd}(\overline{\tau}_0z^m+x_1^{r_2}\overline{\tau}_2z^{m-2}+\cdots+x_1^{r_{m-1}}\overline{\tau}_{m-1}z+x_1^{r_m}\overline{\tau}_m)\end{array}$$

where $\overline{\tau}_0\in\hat{\mathcal{O}}_{V_1,p_1^*}$ is a unit, $\overline{\tau}_i\in\hat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero), for $1\le i\le m-1$ and $\overline{\tau}_m\in\mathfrak{k}[[x_1,\overline{y}_1]]$ with $\mathrm{ord}(\overline{\tau}_m(0,\overline{y}_1))=1$.

We have $\overline{\tau}_0=\overline{\tau}+\Lambda_0$, $\tau_i=\overline{a}_i(x_1,\overline{y}_1)$ for $2\le i\le m-1$, and

$$\overline{\tau}_m=\overline{y}_1+z^{m-1}x_1^{1+s-r_m}\Lambda_1(x_1,\overline{y}_1)+\cdots+x_1^{m+s-r_m}\Lambda_m(x_1,\overline{y}_1)).$$

We thus have the desired form (25).

In the case when p_1 has a form (20), a similar argument to the analysis of (19) shows that p_1^* has a form (26).

Now suppose that p_1 has a form (18). We then have

$$(28)\qquad m_p\mathcal{O}_{U_1,p_1}\subset(x_1y_1,z)\mathcal{O}_{U_1,p_1},$$

unless there exist regular parameters $x_1',y_1'\in\mathcal{O}_{T_1,\Lambda_1(p_1)}$ such that x_1',y_1',z are regular parameters in $\mathcal{O}_{U_1,p_1}$ and

$$(29)\qquad x=x_1',y=(x_1')^ny_1'$$

or

$$(30)\qquad x=x_1'(y_1')^n,y=y_1'$$

for some $n\in\mathbb{N}$. If (29) or (30) holds, then $\hat{\mathcal{O}}_{V_1,p_1^*}=\hat{\mathcal{O}}_{U_1,p_1}$, and (taking $\hat{x}_1=x_1$, $\overline{y}_1=y_1$) we have that a form (24) holds at p_1^*. We may thus assume that (28) holds.

With the notation of Lemma 3.6, we have polynomials φ,ψ such that

$$x=\varphi(\tilde{x}_1,\tilde{y}_1),y=\psi(\tilde{x}_1,\tilde{y}_1)$$

determines the birational extension $\mathcal{O}_{T_0,p_0}\to\mathcal{O}_{T_1,\Lambda_1(p_1)}$, and we have a formal change of variables

$$x_1=\alpha(\tilde{x}_1,\tilde{y}_1)\tilde{x}_1,y_1=\beta(\tilde{x}_1,\tilde{y}_1)\tilde{y}_1$$

for some unit series α and β. We further have expansions

$$a_i(x,y)=x_1^{r_i}y_1^{s_i}\overline{a}_i(x_1,y_1)$$

for $2\le i\le m-1$ where $\overline{a}_i(x_1,y_1)$ are unit series or zero, and

$$a_m(x,y)=x_1^{r_m}y_1^{s_m}\overline{a}_m,$$

where $\overline{a}_m = 0$ or 1. We have $x = \overline{\gamma}x^*$ with $\overline{\gamma} \equiv 1 \bmod m_p^r\hat{\mathcal{O}}_{X,p}$. Set $y^* = y$. At p_1^*, we have regular parameters x_1^*, y_1^* in $\mathcal{O}_{T_1^*,\Lambda_1^*(p_1^*)}$ such that

$$x^* = \varphi(x_1^*, y_1^*), y^* = \psi(x_1^*, y_1^*),$$

and $x_1^*, y_1^*, \tilde{z}$ are regular parameters in $\mathcal{O}_{V_1,\overline{p}_1^*}$ (recall that $z = \sigma\tilde{z}$ in Lemma 3.1). We have regular parameters $\overline{x}_1, \overline{y}_1, \in \hat{\mathcal{O}}_{T_1^*,\Lambda_1^*(p_1^*)}$ defined by

$$\overline{x}_1 = \alpha(x_1^*, y_1^*)x_1^*, \overline{y}_1 = \beta(x_1^*, y_1^*)y_1^*.$$

We calculate

$$u = x^a = (x_1^{a_1}y_1^{b_1})^{t_1} = [\alpha(\tilde{x}_1, \tilde{y}_1)\tilde{x}_1]^{a_1t_1}[\beta(\tilde{x}_1, \tilde{y}_1)\tilde{y}_1]^{b_1t_1}$$

which implies

$$(x^*)^a = [\alpha(x_1^*, y_1^*)x_1^*]^{a_1t_1}[\beta(x_1^*, y_1^*)y_1^*]^{b_1t_1} = \overline{x}_1^{a_1t_1}\overline{y}_1^{b_1t_1}.$$

Set $\hat{x}_1 = \overline{\gamma}^{\frac{a}{a_1t_1}}\overline{x}_1$ to get $u = (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{t_1}$, so that $\hat{x}_1, \overline{y}_1, z$ are permissible parameters at p_1^*.

For $2 \le i \le m$, we have

$$a_i(x, y) = a_i(\overline{\gamma}x^*, y^*) \equiv a_i(x^*, y^*) \bmod m_p^r\hat{\mathcal{O}}_{V,p}$$

and

$$\begin{aligned} a_i(x^*, y^*) &= a_i(\varphi(x_1^*, y_1^*), \psi(x_1^*, y_1^*)) \\ &= \overline{x}_1^{r_i}\overline{y}_1^{s_i}\overline{a}_i(\overline{x}_1, \overline{y}_1) \\ &\equiv \hat{x}_1^{r_i}\overline{y}_1^{s_i}\overline{a}_i(\hat{x}_1, \overline{y}_1) \bmod m_p^r\mathcal{O}_{V_1,p_1^*}. \end{aligned}$$

Thus we have expressions

$$\begin{aligned} u &= (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{t_1} \\ v &= P((\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}}) + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}P_1(\hat{x}_1^{a_1}\overline{y}_1^{b_1}) + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}(\overline{\tau}z^m \\ &\quad + \hat{x}_1^{r_2}\overline{y}_1^{s_2}\overline{a}_2(\hat{x}_1, \overline{y}_1)z^{m-2} + \cdots + \hat{x}_1^{r_m}\overline{y}_1^{s_m}\overline{a}_m + h) \end{aligned} \tag{31}$$

where $\overline{\tau} \in \hat{\mathcal{O}}_{V_1,p_1^*}$ is a unit series and

$$h \in m_p^r\hat{\mathcal{O}}_{V_1,p_1^*} \subset (\hat{x}_1\overline{y}_1, z)^r.$$

Set $s = r - m$, and write

(32)

$$\begin{aligned} h &= z^m\Lambda_0(x_1, \overline{y}_1, z) + z^{m-1}(\hat{x}_1\overline{y}_1)^{1+s}\Lambda_1(\hat{x}_1, \overline{y}_1) + z^{m-2}(\hat{x}_1\overline{y}_1)^{2+s}\Lambda_2(\hat{x}_1, \overline{y}_1) + \cdots \\ &\quad + z(\hat{x}_1\overline{y}_1)^{(m-1)+s}\Lambda_{m-1}(\hat{x}_1, \overline{y}_1) + (\hat{x}_1\overline{y}_1)^{m+s}\Lambda_m(\hat{x}_1, \overline{y}_1) \end{aligned}$$

with $\Lambda_0 \in m_{p_1^*}\hat{\mathcal{O}}_{V_1,p_1^*}$ and $\Lambda_i \in \mathfrak{k}[[\hat{x}_1, \overline{y}_1]]$ for $1 \le i \le m$.

First suppose that $\overline{a}_m = 1$. Substituting into (31), we obtain an expression

$$\begin{aligned} u &= (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{t_1} \\ v &= P((\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}}) + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}P_1(\hat{x}_1^{a_1}\overline{y}_1^{b_1}) \\ &\quad + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}(\overline{\tau}_0z^m + \hat{x}_1^{r_2}\overline{y}_1^{s_2}\overline{\tau}_2z^{m-2} + \cdots + \hat{x}_1^{r_m}\overline{y}_1^{s_m}\overline{\tau}_m) \end{aligned}$$

where $\overline{\tau}_0, \overline{\tau}_m \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units, $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero) for $2 \le i \le m-1$. We have $\overline{\tau}_0 = \overline{\tau} + \Lambda_0$, $\tau_i = \overline{a}_i(\hat{x}_1, \overline{y}_1)$ for $2 \le i \le m-1$, and

$$\overline{\tau}_m = \overline{a}_m + z^{m-1}\hat{x}_1^{1+s-r_m}\overline{y}_1^{1+s-s_m}\Lambda_1(\hat{x}_1, \overline{y}_1) + \cdots + \hat{x}_1^{m+s-r_m}\overline{y}_1^{m+s-s_m}\Lambda_m(\hat{x}_1, \overline{y}_1).$$

We thus have the desired form (24).

Now suppose that $\overline{a}_m = 0$. Then $\overline{a}_{m-1} \neq 0$, and z divides h in (31), so that $\Lambda_m = 0$ in (32). Substituting into (31), we obtain an expression

$$\begin{aligned} u &= (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{t_1} \\ v &= P((\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}) + (\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}P_1(\hat{x}_1^{a_1}\overline{y}_1^{b_1}) \\ &\quad +(\hat{x}_1^{a_1}\overline{y}_1^{b_1})^{\frac{t_1}{a}b}(\overline{\tau}_0 z^m + \hat{x}_1^{r_2}\overline{y}_1^{s_2}\overline{\tau}_2 z^{m-2} + \cdots + \hat{x}_1^{r_{m-1}}\overline{y}_1^{s_{m-1}}\overline{\tau}_{m-1}z) \end{aligned}$$

where $\overline{\tau}_0, \overline{\tau}_{m-1} \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units, $\overline{\tau}_i \in \hat{\mathcal{O}}_{V_1,p_1^*}$ are units (or zero) for $2 \leq i \leq m-2$.

We have $\overline{\tau}_0 = \overline{\tau} + \Lambda_0$, $\tau_i = \overline{a}_i(\hat{x}_1, \overline{y}_1)$ for $2 \leq i \leq m-2$, and

$$\overline{\tau}_{m-1} = \overline{a}_{m-1} + z^{m-1}\hat{x}_1^{1+s-r_{m-1}}\overline{y}_1^{1+s-s_{m-1}}\Lambda_1(\hat{x}_1,\overline{y}_1) + \cdots + \hat{x}_1^{m-1+s-r_{m-1}}\overline{y}_1^{m-1+s-s_{m-1}}\Lambda_{m-1}(\hat{x}_1,\overline{y}_1).$$

We thus have the form (24).

□

Lemma 3.8. *Suppose that X is 2-prepared. Suppose that $p \in X$ is a 1-point with $\sigma_D(p) > 0$ and E is the component of D containing p. Suppose that Y is a finite set of points in X (not containing p). Then there exists an affine neighborhood U of p in X such that*

1) *$Y \cap U = \emptyset$.*
2) *$[E - U \cap E] \cap Sing_1(X)$ is a finite set of points.*
3) *$U \cap D = U \cap E$ and there exists $\overline{x} \in \Gamma(U, \mathcal{O}_X)$ such that $\overline{x} = 0$ is a local equation of E in U.*
4) *There exists an étale map $\pi : U \to \mathbb{A}_k^3 = Spec(\mathfrak{k}[\overline{x}, \overline{y}, \overline{z}])$.*
5) *The Zariski closure C in X of the curve in U with local equations $\overline{x} = \overline{y} = 0$ satisfies the following:*
   i) *C is a nonsingular curve through p.*
   ii) *C contains no 3-points of D.*
   iii) *C intersects 2-curves of D transversally at prepared points.*
   iv) *$C \cap Sing_1(X) \cap (X - U) = \emptyset$.*
   v) *$C \cap Y = \emptyset$.*
   vi) *C intersects $Sing_1(X) - \{p\}$ transversally at general points of curves in $Sing_1(X)$.*
   vii) *There exist permissible parameters x, y, z at p, with $\tilde{x} = \overline{x}, y = \overline{y}$, which satisfy the hypotheses of lemma 3.1.*

Proof. Let H be an effective, very ample divisor on X such that H contains Y and $D - E$, but H does not contain p and does not contain any one dimensional components of $\mathrm{Sing}_1(X, D) \cap E$. There exists $n > 0$ such that $E + nH$ is ample, $\mathcal{O}_X(E+nH)$ is generated by global sections and a general member H' of the linear system $|E+nH|$ does not contain any one dimensional components of $\mathrm{Sing}_1(X, D) \cap E$, and does not contain p. $H + H'$ is ample, so $V = X - (H + H')$ is affine. Further, there exists $f \in \mathfrak{k}(X)$, the function field of X, such that $(f) = H' - (E + nH)$. Thus $\overline{x} = \frac{1}{f} \in \Gamma(V, \mathcal{O}_X)$ as X is normal and $\overline{x}$ has no poles on V. $\overline{x} = 0$ is a local equation of E on V. We have that V satisfies the conclusions 1), 2) and 3) of the lemma.

Let $R = \Gamma(V, \mathcal{O}_X)$. $R = \cup_{s=1}^{\infty}\Gamma(X, \mathcal{O}_X(s(H + H'))$ is a finitely generated $\mathfrak{k}$-algebra. Thus for $s \gg 0$, R is generated by $\Gamma(X, \mathcal{O}_X(s(H + H'))$ as a $\mathfrak{k}$-algebra.

From the exact sequences

$$0 \to \Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p) \to \Gamma(X, \mathcal{O}_X(s(H + H')) \to \mathcal{O}_{X,p}/m_p \cong k$$

and the fact that $1 \in \Gamma(X, \mathcal{O}_X(s(H+H'))$, we have that R is generated by $\Gamma(X, \mathcal{O}_X(s(H+H')) \otimes \mathcal{I}_P)$ as a $\mathfrak{k}$-algebra for all $s \gg 0$.

For $s \gg 0$, and a general member σ of $\Gamma(X, \mathcal{O}_X(s(H+H')) \otimes \mathcal{I}_p)$ we have that the curve $\overline{C} = B \cdot E$, where B is the divisor $B = (\sigma) + s(H+H')$, satisfies the conclusions of 5) of the lemma; since each of the conditions 5i) through 5vii) is an open condition on $\Gamma(X, \mathcal{O}_X(s(H+H') \otimes \mathcal{I}_p))$, we need only establish that each condition holds on a nonempty subset. This follows from the fact that $H+H'$ is ample, Bertini's theorem applied to the base point free linear system $|\varphi^*(s(H+H')) - A|$, where $\varphi : W \to X$ is the blow up of p with exceptional divisor A, and the fact that

$$\varphi_*(\mathcal{O}_W(\varphi^*(s(H+H') - A)) = \mathcal{O}_X(s(H+H')) \otimes \mathcal{I}_p.$$

For fixed $s \gg 0$, let $\overline{x}, \overline{y}_1, \ldots, \overline{y}_n$ be a $\mathfrak{k}$-basis of $\Gamma(X, \mathcal{O}_X(s(H+H')) \otimes \mathcal{I}_p)$, so that $R = \mathfrak{k}[\overline{x}, \overline{y}_1, \ldots, \overline{y}_n]$. We have shown that there exists a Zariski open set $\overline{Z}$ of k^n such that for $(b_1, \ldots, b_n) \in \overline{Z}$, the curve C in X which is the Zariski closure of the curve with local equation $\overline{x} = b_1\overline{y}_1 + \cdots + b_n\overline{y}_n = 0$ in V satisfies 5) of the conclusions of the lemma.

Let $C_1, \ldots, C_t$ be the curves in $\mathrm{Sing}_1(X) \cap V$, and let $p_i \in C_i$ be closed points such that $p, p_1, \ldots, p_t$ are distinct. Let Q_0 be the maximal ideal of p in R, and Q_i be the maximal ideal in R of p_i for $1 \le i \le t$. We have that $\overline{x}$ is nonzero in Q_i/Q_i^2 for all i. For a matrix $A = (a_{ij}) \in \mathfrak{k}^{2n}$, and $1 \le i \le 2$, let

$$L_i^A(\overline{y}_1, \ldots, \overline{y}_n) = \sum_{j=1}^n a_{ij}\overline{y}_j.$$

There exist $\alpha_{jk} \in \mathfrak{k}$ such that $Q_k = (\overline{y}_1 - \alpha_{1,k}, \ldots, \overline{y}_n - \alpha_{n,k})$ for $0 \le k \le t$. By our construction, we have $\alpha_{1,0} = \cdots = \alpha_{n,0} = 0$. For each $0 \le k \le t$, there exists a non empty Zariski open subset Z_k of k^{2n} such that

$$\overline{x}, L_1^A(\overline{y}_1, \ldots, \overline{y}_n) - L_1^A(\alpha_{1,k}, \ldots, \alpha_{n,k}), L_2^A(\overline{y}_1, \ldots, \overline{y}_n) - L_2^A(\alpha_{1,k}, \ldots, \alpha_{n,k})$$

is a $\mathfrak{k}$-basis of Q_k/Q_{k+1}^2. Suppose $(a_{1,1}, \ldots, a_{1,n}) \in \overline{Z}$ and $A \in Z_0 \cap \cdots \cap Z_t$.

We will show that $\overline{x}, L_1^A, L_2^A$ are algebraically independent over $\mathfrak{k}$. Suppose not. Then there exists a nonzero polynomial $h \in \mathfrak{k}[t_1, t_2, t_3]$ such that $h(\overline{x}, L_1^A, L_2^A) = 0$. Write $h = H + h'$ where H is the leading form of h, and $h' = h - H$ is a polynomial of larger order than the degree r of H. Now $H(\overline{x}, L_1^A, L_2^A) = -h'(\overline{x}, L_1^A, L_2^A)$, so that $H(\overline{x}, L_1^A, L_2^A) = 0$ in Q_0^r/Q_0^{r+1}. Thus $H = 0$, since R_{Q_0} is a regular local ring, which is a contradiction. Thus $\overline{x}, L_1^A, L_2^A$ are algebraically independent. Without loss of generality, we may assume that $L_i^A = \overline{y}_i$ for $1 \le i \le 2$.

Let $S = \mathfrak{k}[\overline{x}, \overline{y}_1, \overline{y}_2]$, a polynomial ring in 3 variables over $\mathfrak{k}$. $S \to R$ is unramified at Q_i for $0 \le i \le t$ since

$$(\overline{x}, \overline{y}_1 - \alpha_{1,i}, \overline{y}_2 - \alpha_{2,i})R_{Q_i} = Q_iR_{Q_i}$$

for $0 \le i \le t$.

Let W be the closed locus in V where $V \to \mathrm{Spec}(S)$ is not étale. We have that $p, p_1, \ldots, p_t \notin W$, so there exists an ample effective divisor $\overline{H}$ on X such that $W \subset \overline{H}$ and $p, p_1, \ldots, p_t \notin \overline{H}$. Let $U = V - \overline{H}$. U is affine, and $U \to \mathrm{Spec}(S) \cong \mathbb{A}^3$ is étale, so satisfies 4) of the conclusions of the lemma.

□

Lemma 3.9. *Suppose X is 2-prepared with respect to $f : X \to S$, $p \in D$ is a prepared point, and $\pi_1 : X_1 \to X$ is the blow up of p. Then all points of $\pi_1^{-1}(p)$ are prepared.*

Proof. The conclusions follow from substitution of local equations of the blow up of a point into a prepared form (1), (2) or (3). □

Lemma 3.10. *Suppose that X is 2-prepared with respect to $f : X \to S$, and that C is a permissible curve for D, which is not a 2-curve. Suppose that $p \in C$ satisfies $\sigma_D(p) = 0$. Then there exist permissible parameters x, y, z at p such that one of the following forms hold:*

1) *p is a 1-point of D of the form of (1), $F = z$ and $x = y = 0$ are formal local equations of C at p.*
2) *p is a 1-point of D of the form of (1), $F = z$ and $x = z = 0$ are formal local equations of C at p.*
3) *p is a 1-point of D of the form of (1), $F = z$, $x = z + y^r\sigma(y) = 0$ are formal local equations of C at p, where $r > 1$ and σ is a unit series.*
4) *p is a 2-point of D of the form of (2), $F = z$, $x = z = 0$ are formal local equations of C at p.*
5) *p is a 2-point of D of the form of (2), $F = z$, $x = f(y, z) = 0$ are formal local equations of C at p, where $f(y, z)$ is not divisible by z.*
6) *p is a 2-point of D of the form of (2), $F = 1$ (so that $ad - bc \neq 0$) and $x = z = 0$ are formal local equations of C at p.*

Further, there are at most a finite number of 1-points on C satisfying condition 3) (and not satisfying condition 1) or 2)).

Proof. Suppose that p is a 1-point. We have permissible parameters x, y, z at p such that a form (1) holds at p with $F = z$. There exists a series $f(y, z)$ such that $x = f = 0$ are formal local equations of C at p. By the formal implicit function theorem, we get one of the forms 1), 2) or 3). A similar argument shows that one of the forms 4), 5) or 6) must hold if p is a 2-point.

Now suppose that $p \in C$ is a 1-point, $\sigma_D(p) = 0$ and a form 3) holds at p. There exist permissible parameters x, y, z at p, with an expression (1), such that $x = z = 0$ are formal local equations of C at p and x, y, z are uniformizing parameters on an étale cover U of an neighborhood of p, where we can choose U so that

$$\frac{\partial F}{\partial y} = \frac{1}{x^b}\frac{\partial v}{\partial y} \in \Gamma(U, \mathcal{O}_X).$$

Since there is not a form 2) at p, we have that z does not divide $F(0, y, z)$, so that $F(0, y, 0) \neq 0$. Since F has no constant term, we have that $\frac{\partial F}{\partial y}(0, y, 0) \neq 0$. There exists a Zariski open subset of $\mathfrak{k}$ such that $\alpha \in \mathfrak{k}$ implies $x, y - \alpha, z$ are regular parameters at a point $q \in U$. There exists a Zariski open subset of $\mathfrak{k}$ of such α so that $\frac{\partial F}{\partial y}(0, \alpha, 0) \neq 0$. Thus $x, y - \alpha, z$ are permissible parameters at q giving a form 1) at $q \in C$.

□

Lemma 3.11. *Suppose that X is 2-prepared. Suppose that C is a permissible curve on X which is not a 2-curve and $p \in C$ satisfies $\sigma_D(p) = 0$. Further suppose that either a form 3) or 5) of the conclusions of Lemma 3.10 hold at p. Then there exists a sequence of blow ups of points $\pi_1 : X_1 \to X$ above p such that X_1 is 2-prepared and $\sigma_{D_1}(p_1) = 0$ for all $p_1 \in \pi_1^{-1}(p)$, and the strict transform of C on X_1 is permissible, and has the form 4) or 6) of Lemma 3.10 at the point above p.*

Proof. If p is a 1-point, let $\pi' : X' \to X$ be the blow ups of p, and let C' be the strict transform of C on X'. Let p' be the point on C' above p. Then p' is a 2-point and $\sigma_D(p') = 0$. We may thus assume that p is a 2-point and a form 5) holds at p. For $r \in \mathbb{Z}_+$, let

$$X_r \to X_{r-1} \to \cdots \to X_1 \to X$$

be the sequence of blow ups of the point p_i which is the intersection of the strict transform C_i of C on X_i with the preimage of p.

There exist permissible parameters x,y,z at p such that $x=z=0$ are formal local equations of C at p, and a form (2) holds at p with $F=x\Omega+f(y,z)$. We have that $\operatorname{ord} f(y,z)=1$, $\operatorname{ord}\Omega(0,y,z)\ge 1$, y does not divide $f(y,z)$ and z does not divide $f(y,z)$.

At p_r, we have permissible parameters x_r,y_r,z_r such that

$$x=x_ry_r^r,\ y=y_r,\ z=z_ry_r^r.$$

$x_r=z_r=0$ are local equations of C_r at p_r. We have a form (2) at p_r with

$$\begin{array}{lll} u &=& (x_r^ay_r^{ar+b})^l\\ v &=& P(x_r^ay_r^{ar+b})+x_r^cy_r^{cr+d+r}F' \end{array}$$

where

$$F'=x_r\Omega+\frac{f(y_r,z_ry_r^r)}{y_r^r},$$

if $\frac{f(y_{r-1},z_{r-1}y_{r-1}^{r-1})}{y_{r-1}^{r-1}}$ is not a unit series. Thus for r sufficiently large, we have that F' is a unit, so that a form 6) holds at p_r.

□

Lemma 3.12. *Suppose that X is 2-prepared and that C_1 is a permissible curve on X. Suppose that $q\in C$ is a point with $\sigma_D(q)=0$ which has a form 1), 4) or 6) of Lemma 3.10. Let $\pi_1:X_1\to X$ be the blow up of C. Then X_1 is 3-prepared in a neighborhood of $\pi_1^{-1}(q)$. Further, $\sigma_{D_1}(q_1)=0$ for all $q_1\in\pi_1^{-1}(q)$.*

Proof. The conclusions follow from substitution of local equations of the blow up of C into the forms 1), 4) and 6) of Lemma 3.10. □

Proposition 3.13. *Suppose that X is 2-prepared. Then there exists a sequence of permissible blow ups $\pi_1:X_1\to X$, such that X_1 is 3-prepared. We further have that $\sigma_D(p_1)\le\sigma_D(p)$ for all $p\in X$ and $p_1\in\pi_1^{-1}(p)$.*

Proof. Let T be the points $p\in X$ such that X is not 3-prepared at p. By Lemmas 3.4 and 2.5, after we perform a sequence of blow ups of 2-curves, we may assume that T is a finite set consisting of 1-points of D.

Suppose that $p\in T$. Let $T'=T\setminus\{p\}$. Let $U=\operatorname{Spec}(R)$ be the affine neighborhood of p in X and let C be the curve in X of the conclusions of Lemma 3.8 (with $Y=T'$), so that C has local equations $\overline{x}=\overline{y}=0$ in U.

Let $\Sigma_1=C\cap\operatorname{Sing}_1(X)$. $\Sigma_1=\{p=p_0,\ldots,p_r\}$ is the union of $\{p\}$ and a finite set of general points of curves in $\operatorname{Sing}_1(X)$, which must be 1-points. We have that $\Sigma_1\subset U$. Let

$$\Sigma_2=\{q\in C\cap U\mid \sigma_D(q)=0\text{ and a form 2) of Lemma 3.10 holds at }q\}.$$

Σ_2 is a finite set by Lemma 3.10. Let $\Sigma_3=C\setminus U$, a finite set of 1-points and 2-points which are prepared.

Set $U'=U\setminus\Sigma_2$. There exists a unit $\tau\in R$ and $a\in\mathbb{Z}_+$ such that $u=\tau\overline{x}^a$.

By 5 vi), 5 vii) of Lemma 3.8 and Lemma 3.2, there exist $z_i\in\hat{\mathcal{O}}_{X,p_i}$ such that for all $p_i\in\Sigma_1$, $x=\tau^{\frac{1}{a}}\overline{x},\overline{y},z_i$ are permissible parameters at p_i giving a form (9).

Let $t=\max\{r(p_i)\mid 0\le i\le r\}$, where $r(p_i)$ are calculated from (23)) of Lemma 3.7. There exists $\lambda\in R$ such that $\lambda\equiv\tau^{-\frac{1}{a}}\bmod m_{p_i}^t\hat{\mathcal{O}}_{X,p_i}$ for $0\le i\le r$. Let $x^*=\lambda^{-1}\overline{x}$, $\overline{\gamma}=\tau^{\frac{1}{a}}\lambda$. Then $x=\tau^{\frac{1}{a}}\overline{x}=\overline{\gamma}x^*$ with $\overline{\gamma}\equiv 1\bmod m_{p_i}^t\hat{\mathcal{O}}_{X,p_i}$ for $0\le i\le r$. Let $U'=U\setminus\Sigma_2$.

Let $T_0^* = \mathrm{Spec}(\mathfrak{k}[x^*, \overline{y}])$, and let $T_1^* \to T_0^*$ be a sequence of blow ups of points above $(x^*, \overline{y})$ such that the conclusions of Lemma 3.7 hold on $U_1' = U' \times_{T_0^*} T_1^*$ above all p_i with $0 \le i \le r$. The projection $\lambda_1 : U_1' \to U'$ is a sequence of blow ups of sections over C. λ_1 is permissible and $\lambda_1^{-1}(C \cap (U' \setminus \Sigma_1))$ is prepared by Lemma 3.12.

All points of $\Sigma_2 \cup \Sigma_3$ are prepared. Thus by Lemma 3.9, Lemmas 3.11 and Lemma 3.12, by interchanging some blowups of points above $\Sigma_2 \cup \Sigma_4$ between blow ups of sections over C, we may extend λ_1 to a sequence of permissible blow ups over X to obtain the desired sequence of permissible blow ups $\pi_1 : X_1 \to X$ such that X_1 is 2-prepared. π_1 is an isomorphism over T', X_1 is 3-prepared over $\pi_1^{-1}(X_1 \setminus T')$, and $\sigma_D(p_1) \le \sigma_D(p)$ for all $p \in X_1 \setminus T'$.

By induction on $|T|$, we may iterate this procedure a finite number of times to obtain the conclusions of Proposition 3.13.

□

The following proposition is proven in a similar way.

Proposition 3.14. *Suppose that X is 1-prepared and D' is a union of irreducible components of D. Suppose that there exists a neighborhood V of D' such that V is 2-prepared and V is 3-prepared at all 2-points and 3-points of V.*

Let A be a finite set of 1-points of D', such that A is contained in $Sing_1(X)$ and A contains the points where V is not 3-prepared, and let B be a finite set of 2-points of D'. Then there exists a sequence of permissible blow ups $\pi_1 : X_1 \to X$ such that

1) *X_1 is 3-prepared in a neighborhood of $\pi_1^{-1}(D')$.*
2) *π_1 is an isomorphism over $X_1 \setminus D'$.*
3) *π_1 is an isomorphism in a neighborhood of B.*
4) *π_1 is an isomorphism over generic points of 2-curves on D' and over 3-points of D'.*
5) *Points on the intersection of the strict transform of D' on X_1 with $\pi_1^{-1}(A)$ are 2-points of D_{X_1}.*
6) *$\sigma_D(p_1) \le \sigma_D(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.*

4. Reduction of σ_D above a 3-prepared point.

Theorem 4.1. *Suppose that $p \in X$ is a 1-point such that X is 3-prepared at p, and $\sigma_D(p) > 0$. Let x, y, z be permissible parameters at p giving a form (14) at p. Let U be an étale cover of an affine neighborhood of p in which x, y, z are uniformizing parameters. Then $xz = 0$ gives a toroidal structure $\overline{D}$ on U. Let I be the ideal in $\Gamma(U, \mathcal{O}_X)$ generated by z^m, x^{r_m} if $\tau_m \ne 0$, and by*

$$\{x^{r_i} z^{m-i} \mid 2 \le i \le m-1 \text{ and } \tau_i \ne 0\}.$$

Suppose that $\psi : U' \to U$ is a toroidal morphism with respect to $\overline{D}$ such that U' is nonsingular and $I\mathcal{O}_{U'}$ is locally principal. Then (after possibly replacing U with a smaller neighborhood of p) U' is 2-prepared and $\sigma_D(q) < \sigma_D(p)$ for all $q \in U'$.

There is (after possibly replacing U with a smaller neighborhood of p) a unique, minimal toroidal morphism $\psi : U' \to U$ with respect to $\overline{D}$ with has the property that U' is nonsingular, 2-prepared and $\Gamma_D(U') < \sigma_D(p)$. This map ψ factors as a sequence of permissible blowups $\pi_i : U_i \to U_{i-1}$ of sections C_i over the two curve C of $\overline{D}$. U_i is 1-prepared for $U_i \to S$. We have that the curve C_i blown up in $U_{i+1} \to U_i$ is in $Sing_{\sigma_D(p)}(U_i)$ if C_i is not a 2-curve of D_{U_i}, and that C_i is in $Sing_1(U_i)$ if C_i is a 2-curve of D_{U_i}.

Proof. Suppose that $\psi : U' \to U$ is toroidal for $\overline{D}$ and U' is nonsingular. Let $\overline{D}' = \psi^{-1}(\overline{D})$.

The set of 2-curves of $\overline{D}'$ is the disjoint union of the 2-curves of $D_{U'}$ and the 2-curve which is the intersection of the strict transform of the surface $z = 0$ on U' with $D_{U'}$. ψ factors as a sequence of blow ups of 2-curves of (the preimage of) $\overline{D}$. We will verify the following three statements, from which the conclusions of the theorem follow.

(33) If $q \in \psi^{-1}(p)$ and $I\mathcal{O}_{U',q}$ is principal, then $\sigma_D(q) < \sigma_D(p)$.
In particular, $\sigma_D(q) < \sigma_D(p)$ if q is a 1-point of $\overline{D}'$.

(34) If C' is a 2-curve of $D_{U'}$, then U' is prepared at $q = C' \cap \psi^{-1}(p)$
if and only if $\sigma_D(q) < \infty$
if and only if $I\mathcal{O}_{U',q}$ is principal
if and only if U' is prepared at all $q' \in C'$ in a neighborhood of q.

(35)
If C' is the 2-curve of $\overline{D}'$ which is the intersection of $D_{U'}$ with the strict transform of $\tilde{z} = 0$ in U', then $\sigma_D(q) \le \sigma_D(p)$ if $q = C' \cap \psi^{-1}(p)$, and $\sigma_D(q') = \sigma_D(q)$ for $q' \in C'$ in a neighborhood of q.

Suppose that $q \in \psi^{-1}(p)$ is a 1-point for $\overline{D}'$. Then $I\hat{\mathcal{O}}_{U',q}$ is principal. At q, we have permissible parameters x_1, y, z_1 defined by

$$x = x_1^{a_1}, z = x_1^{b_1}(z_1 + \alpha) \tag{36}$$

for some $a_1, b_1 \in \mathbb{Z}_+$ and $0 \ne \alpha \in \mathfrak{k}$. Substituting into (14), we have

$$u = x_1^{aa_1}, v = P(x_1^{a_1}) + x_1^{ba_1} G$$

where

$$G = \tau_0 x_1^{b_1 m}(z_1+\alpha)^m + \tau_2 x_1^{a_1 r_2 + b_1(m-2)}(z_1+\alpha)^{m-2} + \cdots + \tau_{m-1} x_1^{a_1 r_{m-1} + b_1}(z_1+\alpha) + \tau_m x_1^{a_1 r_m}.$$

Let x_1^s be a local generator of $I\hat{\mathcal{O}}_{U',q}$. Let $G' = \frac{G}{x_1^s}$.

If z^m is a local generator of $I\hat{\mathcal{O}}_{U',q}$, then G' has an expansion

$$G' = \tau'(z_1+\alpha)^m + g_2(z_1+\alpha)^{m-2} + \cdots + g_{m-1}(z_1+\alpha) + g_m + x_1\Omega_1 + y\Omega_2$$

where $0 \ne \tau' = \tau(0,0,0) \in \mathfrak{k}$, $g_2, \ldots, g_m \in \mathfrak{k}$ and $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U',q}$. We have $\mathrm{ord}(G'(0,0,z_1)) \le m-1$. Setting $F' = G' - G'(x_1,0,0)$ and $P'(x_1) = P(x_1^{a_1}) + x_1^{ba_1 + b_1 m} G'(x_1,0,0)$, we have an expression

$$u = x_1^{aa_1}, v = P'(x_1) + x_1^{ba_1 + b_1 m} F'$$

of the form of (1). Thus U' is 2-prepared at q with $\sigma_{D'}(q) < m - 1 = \sigma_D(p)$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U',q}$, but there exists some i with $2 \le i \le m-1$ such that $x^{r_i} z^{m-i}$ is a local generator of $I\hat{\mathcal{O}}_{U',q}$. Let h be the smallest i with this property. Then G' has an expression

$$G' = g_h(z_1+\alpha)^{m-h} + \cdots + g_m + x_1\Omega_1 + y_1\Omega_2$$

for some $g_i \in \mathfrak{k}$ with $g_h \ne 0$ and $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U',q}$. As in the previous case, we have that U' is 2-prepared at q with $\sigma_D(q) < m - h - 1 < m - 1 = \sigma_D(p)$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U',q}$ and $x^{r_i}z^{m-i}$ is not a local generator of $I\hat{\mathcal{O}}_{U',q}$ for $2 \le i \le m-1$. Then $x_1^{r_m}$ is a local generator of $I\mathcal{O}_{U',q}$, and we have an expression

$$G' = \Lambda + x_1\Omega_1,$$

where $\Lambda(x_1, y, z_1) = \tau_m(x_1^{a_1}, y, x_1^{b_1}(z_1 + \alpha))$ and $\Omega_1 \in \hat{\mathcal{O}}_{U',q}$. Then

$$\text{ord } \Lambda(0, y, 0) = \text{ord } \tau_m(0, y, 0) = 1,$$

and we have that U' is prepared at q.

Now suppose that $q \in \psi^{-1}(p)$ is a 2-point for $D_{U'}$. We have permissible parameters x_1, y, z_1 in $\hat{\mathcal{O}}_{U',q}$ such that

$$x = x_1^{a_1}z_1^{b_1}, z = x_1^{c_1}z_1^{d_1} \tag{37}$$

with $a_1, b_1 > 0$ and $a_1d_1 - b_1c_1 = \pm 1$. Substituting into (14), we have

$$u = x_1^{a_1a}z_1^{b_1a}, v = P(x_1^{a_1}z_1^{b_1}) + x_1^{a_1b}z_1^{b_1b}G$$

where

$$G = \tau_0 x_1^{c_1m}z_1^{d_1m} + \tau_2 x_1^{r_2a_1 + c_1(m-2)}z_1^{r_2b_1 + d_1(m-2)} + \cdots + \tau_{m-1}x_1^{a_1r_{m-1}+c_1}z_1^{b_1r_{m-1}+d_1} + \tau_m x_1^{a_1r_m}z_1^{b_1r_m}.$$

Let C' be the 2-curve of $D_{U'}$ containing q. Since ord $(\tau_m(0, y, 0)) = 1$ (if $\tau_m \ne 0$) we see that the three statements $\sigma_D(q) < \infty$, $\sigma_D(q) = 0$ and $I\mathcal{O}_{U',q}$ is principal are equivalent. Further, we have that $\sigma_D(q') = \sigma_D(q)$ for $q' \in C'$ in a neighborhood of q.

Suppose that $I\mathcal{O}_{U',q}$ is principal and let $x_1^sz_1^t$ be a local generator of $I\hat{\mathcal{O}}_{U',q}$. Let $G' = G/x_1^sz_1^t$. We have that

$$u = (x_1^{a_1}z_1^{b_1})^a,\ v = P(x_1^{a_1}z_1^{b_1}) + x_1^{a_1b+s}z_1^{bb_1+t}G'$$

has the form (2), since we have made a monomial substitution in x and z. If z^m or $x^{r_i}z^{m-i}$ for some $i < m$ is a local generator of $I\hat{\mathcal{O}}_{U',q}$, then G' is a unit in $\hat{\mathcal{O}}_{U',q}$. If none of z^m, $x^{r_i}z^{m-i}$ for $i < m$ are local generators of $I\hat{\mathcal{O}}_{U',q}$, then

$$G' = \Lambda + x_1\Omega_1 + z_1\Omega_2,$$

where

$$\Lambda(x_1, y_1, z_1) = \tau_m(x_1^{a_1}z_1^{b_1}, y, x_1^{c_1}z_1^{d_1})$$

and $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U',q}$. Thus

$$\text{ord } \Lambda(0, y, 0) = \text{ord } \tau_m(0, y, 0) = 1.$$

We thus have that U' is prepared at q.

The final case is when $q \in \psi^{-1}(p)$ is on the 2-curve C' of $\overline{D}'$ which is the intersection of $D_{U'}$ with the strict transform of $z = 0$ in U'. Then there exist permissible parameters x_1, y, z_1 at q such that

$$x = x_1, z = x_1^{b_1}z_1 \tag{38}$$

for some $b_1 \in \mathbb{Z}_+$. The equations $x_1 = z_1 = 0$ are local equations of C' at q. Let

$$s = \min\{b_1m, r_i + b_1(m-i) \text{ with } \tau_i \ne 0 \text{ for } 2 \le i \le m-1, r_m \text{ if } \tau_i \ne 0\}.$$

We have an expression of the form (1) at q,

$$\begin{aligned} u &= x_1^a \\ v &= P(x_1^a) + x_1^{ab+s}G' \end{aligned}$$

with

$$G' = \tau_0 x_1^{b_1 m - s} z_1^m + \tau_2 x_1^{r_2 + b_1(m-2) - s} z_1^{m-2} + \cdots + \tau_{m-1} x_1^{r_{m-1} + b_1 - s} z_1 + \tau_m x_1^{r_m - s}.$$

We see that $\sigma_D(q) \le \sigma_D(p)$ (with $\sigma_D(q) < \sigma_D(p)$ if $s = r_i + b_1(m-i)$ for some i with $2 \le i \le m-1$ or $s = r_m$) and $\sigma_D(q') = \sigma_D(q)$ for q' in a neighborhood of q on C'.

Suppose that $I\mathcal{O}_{U',q}$ is principal. Then x^{r_m} generates $I\hat{\mathcal{O}}_{U',q}$. We have that $G' = x_1^{r_m}\Omega$ where $\Omega \in \hat{\mathcal{O}}_{U',q}$ satisfies ord $\Omega(0,y,0) = 1$. Thus U' is prepared at q.

$\square$

We will now construct the function $\omega(m, r_2, \ldots, r_{m-1})$ where $m > 1$, $r_i \in \mathbb{N}$ for $2 \le i \le m-1$ and $r_{m-1} > 0$.

Let I be the ideal in the polynomial ring $\mathfrak{k}[x,z]$ generated by z^m and $x^{r_i} z^{m-i}$ for all i such that $2 \le i \le m-1$ and $r_i > 0$. Let $\mathfrak{m} = (x,z)$ be the maximal ideal of $k[x,z]$. Let $\Phi : V_1 \to V = \mathrm{Spec}(\mathfrak{k}[x,z])$ be the toroidal morphism with respect to the divisor $xz = 0$ on V such that V_1 is the minimal nonsingular surface such that

1) $I\mathcal{O}_{V_1,q}$ is principal if $q \in \Phi^{-1}(\mathfrak{m})$ is not on the strict transform of $z = 0$.
2) If q is the intersection point of the strict transform of $z = 0$ and $\Phi^{-1}(\mathfrak{m})$, so that q has regular parameters x_1, z_1, with $x = x_1, z = x_1^b z_1$ for some $b \in \mathbb{Z}_+$, then $r_i + b_1(m-i) < b_1 m$ for some $2 \le i \le m-1$ with $r_i > 0$.

Every $q \in \Phi^{-1}(\mathfrak{m})$ which is not on the strict transform of $z = 0$ has regular parameters x_1, z_1 at q which are related to x, z by one of the following expressions:

$$x = x_1^{a_1},\ z = x_1^{b_1}(z_1 + \alpha) \tag{39}$$

for some $0 \ne \alpha \in \mathfrak{k}$ and $a_1, b_1 > 0$, or

$$x = x_1^{a_1} z_1^{b_1},\ z = x_1^{c_1} z_1^{d_1} \tag{40}$$

with $a_1, b_1 > 0$ and $a_1 d_1 - b_1 c_1 = \pm 1$. There are only finitely many values of a_1, b_1 occurring in expressions (39), and a_1, b_1, c_1, d_1 occurring in expressions (40).

The point q on the intersection of the strict transform of $z = 0$ and $\Phi^{-1}(\mathfrak{m})$ has regular parameters x_1, z_1 defined by

$$x = x_1,\ z = x_1^{b_1} z_1 \tag{41}$$

for some $b_1 > 0$.

Now we define $\omega = \omega(m, r_2, \ldots, r_{m-1})$ to be a number such that

$$\omega > \max\{\frac{b_1}{a_1}m, r_i + \frac{b_1}{a_1}(m-i) \text{ for } 2 \le i \le m-1 \text{ such that } r_i > 0\}.$$

For all expressions (39),

$$\omega > \max\{\frac{c_1}{a_1}m, \frac{d_1}{b_1}m, r_i + \frac{c_1}{a_1}(m-i), r_i + \frac{d_1}{b_1}(m-i) \text{ for } 2 \le i \le m-1 \text{ such that } r_i > 0\}$$

for all expressions (40), and

$$\omega > \max\{b_1 m, r_i + b_1(m-i) \text{ for } 2 \le i \le m-1 \text{ such that } r_i > 0\}$$

in (41).

Theorem 4.2. *Suppose that $p \in$ $Sing_1(X)$ is a 1-point and X is 3-prepared at p. Let x, y, z be permissible parameters at p giving a form (15) at p. Let U be an étale cover of an affine neighborhood of p in which x, y, z are uniformizing parameters. Then $xz = 0$ gives a toroidal structure $\overline{D}$ on U.*

There is (after possibly replacing U with a smaller neighborhood of p) a unique, minimal toroidal morphism $\psi: U' \rightarrow U$ with respect to $\overline{D}$ with has the property that U' is nonsingular, 2-prepared and $\Gamma_D(U') < \sigma_D(p)$. This map ψ factors as a sequence of permissible blowups $\pi_i : U_i \rightarrow U_{i-1}$ of sections C_i over the two curve C of $\overline{D}$. U_i is 1-prepared for $U_i \rightarrow S$. We have that the curve C_i blown up in $U_{i+1} \rightarrow U_i$ is in $Sing_{\sigma_D(p)}(U_i)$ if C_i is not a 2-curve of D_{U_i}, and that C_i is in $Sing_1(U_i)$ if C_i is a 2-curve of D_{U_i}.

Proof. The proof is similar to that of Theorem 4.1, using the fact that $t > \omega(m, r_2, \ldots, r_{m-1})$ as defined above. $\square$

Theorem 4.3. *Suppose that $p \in X$ is a 2-point and X is 3-prepared at p with $\sigma_D(p) > 0$. Let x, y, z be permissible parameters at p giving a form (13) at p. Let U be an étale cover of an affine neighborhood of p in which x, y, z are uniformizing parameters on U. Then $xyz = 0$ gives a toroidal structure $\overline{D}$ on U. Let I be the ideal in $\Gamma(U, \mathcal{O}_X)$ generated by z^m, $x^{r_m}y^{s_m}$ if $\tau_m \neq 0$ and*

$$\{x^{r_i}y^{s_i}z^{m-i} \mid 2 \leq i \leq m-1 \text{ and } \tau_i \neq 0\}.$$

Suppose that $\psi : U_1 \rightarrow U$ is a toroidal morphism with respect to $\overline{D}$ such that U_1 is nonsingular and $I\mathcal{O}_{U_1}$ is locally principal. Then (after possibly replacing U with a smaller neighborhood of p) U_1 is 2-prepared for $U_1 \rightarrow S$, with $\sigma_D(q) < \sigma_D(p)$ for all $q \in U_1$.

Proof. Suppose that $q \in \psi^{-1}(p)$ is a 1-point for $\psi^{-1}(\overline{D})$. Then q is also a 1-point for D_{U_1}. Since ψ is toroidal with respect to $\overline{D}$, there exist regular parameters $\hat{x}_1, \hat{y}_1, \hat{z}_1$ in $\hat{\mathcal{O}}_{X_1,q}$ and a matrix $A = (a_{ij})$ with nonegative integers as coefficients such that Det $A = \pm 1$, and we have an expression

$$\begin{array}{rcl} x & = & \hat{x}_1^{a_{11}}(\hat{y}_1+\alpha)^{a_{12}}(\hat{z}_1+\beta)^{a_{13}} \\ y & = & \hat{x}_1^{a_{21}}(\hat{y}_1+\alpha)^{a_{22}}(\hat{z}_1+\beta)^{a_{23}} \\ z & = & \hat{x}_1^{a_{31}}(\hat{y}_1+\alpha)^{a_{32}}(\hat{z}_1+\beta)^{a_{33}} \end{array} \tag{42}$$

with $a_{11}, a_{21}, a_{31} \neq 0$ and $0 \neq \alpha, \beta \in \mathfrak{k}$. Set

$$\overline{x}_1 = \hat{x}_1(\hat{y}_1+\alpha)^{\frac{a_{12}}{a_{11}}}(\hat{z}_1+\beta)^{\frac{a_{13}}{a_{11}}} \in \hat{\mathcal{O}}_{X_1,q}.$$

Substituting into (42), we have

$$\begin{array}{rcl} x & = & \overline{x}_1^{a_{11}} \\ y & = & \overline{x}_1^{a_{21}}(\hat{y}_1+\alpha)^{a_{22}-\frac{a_{21}a_{12}}{a_{11}}}(\hat{z}_1+\beta)^{a_{23}-\frac{a_{21}a_{13}}{a_{11}}} \\ z & = & \overline{x}_1^{a_{31}}(\hat{y}_1+\alpha)^{a_{32}-\frac{a_{31}a_{12}}{a_{11}}}(\hat{z}_1+\beta)^{a_{33}-\frac{a_{31}a_{13}}{a_{11}}}. \end{array} \tag{43}$$

Let $B = (b_{ij})$ be the adjoint matrix of A. Let $\overline{\alpha} = \alpha^{\frac{b_{33}}{a_{11}}}\beta^{-\frac{b_{23}}{a_{11}}}$, $\overline{\beta} = \alpha^{-\frac{b_{32}}{a_{11}}}\beta^{\frac{b_{22}}{a_{11}}}$. Set

$$\overline{y}_1 = \frac{y}{\overline{x}_1^{a_{21}}} - \overline{\alpha}, \overline{z}_1 = \frac{z}{\overline{x}_1^{a_{31}}} - \overline{\beta}.$$

We will show that $\overline{x}_1, \overline{y}_1, \overline{z}_1$ are regular parameters in $\hat{\mathcal{O}}_{X_1,q}$. We have that

$$\begin{array}{rcl} (\hat{y}_1+\alpha)^{a_{22}-\frac{a_{21}a_{12}}{a_{11}}}(\hat{z}_1+\beta)^{a_{23}-\frac{a_{21}a_{13}}{a_{11}}} & = & \overline{\alpha} + \frac{b_{33}}{a_{11}}\alpha^{\frac{b_{33}}{a_{11}}-1}\beta^{-\frac{b_{23}}{a_{11}}}\hat{y}_1 - \frac{b_{23}}{a_{11}}\alpha^{\frac{b_{33}}{a_{11}}}\beta^{-\frac{b_{23}}{a_{11}}-1}\hat{z}_1 + \cdots \\ (\hat{y}_1+\alpha)^{a_{32}-\frac{a_{31}a_{12}}{a_{11}}}(\hat{z}_1+\beta)^{a_{33}-\frac{a_{31}a_{13}}{a_{11}}} & = & \overline{\beta} - \frac{b_{32}}{a_{11}}\alpha^{-\frac{b_{32}}{a_{11}}-1}\beta^{\frac{b_{22}}{a_{11}}}\hat{y}_1 + \frac{b_{22}}{a_{11}}\alpha^{-\frac{b_{32}}{a_{11}}}\beta^{\frac{b_{22}}{a_{11}}-1}\hat{z}_1 + \cdots \end{array}$$

Let

$$C = \begin{pmatrix} \frac{b_{33}}{a_{11}}\alpha^{\frac{b_{33}}{a_{11}}-1}\beta^{-\frac{b_{23}}{a_{11}}} & -\frac{b_{23}}{a_{11}}\alpha^{\frac{b_{33}}{a_{11}}}\beta^{-\frac{b_{23}}{a_{11}}-1} \\ -\frac{b_{32}}{a_{11}}\alpha^{-\frac{b_{32}}{a_{11}}-1}\beta^{\frac{b_{22}}{a_{11}}} & \frac{b_{22}}{a_{11}}\alpha^{-\frac{b_{32}}{a_{11}}}\beta^{\frac{b_{22}}{a_{11}}-1} \end{pmatrix}.$$

We must show that C has rank 2. C has the same rank as

$$\begin{pmatrix} b_{33}\beta & -b_{23}\alpha \\ b_{32}\beta & -b_{22}\alpha \end{pmatrix} = \begin{pmatrix} b_{33} & b_{23} \\ b_{32} & b_{22} \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & -\alpha \end{pmatrix}.$$

Since $\alpha, \beta \neq 0$, C has the same rank as

$$B' = \begin{pmatrix} b_{33} & b_{23} \\ b_{32} & b_{22} \end{pmatrix}.$$

Since B has rank 3,

$$\begin{pmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

has rank 2. Since

$$\begin{pmatrix} b_{21} \\ b_{31} \end{pmatrix} = -\frac{a_{21}}{a_{11}} \begin{pmatrix} b_{22} \\ b_{32} \end{pmatrix} + \frac{a_{31}}{a_{11}} \begin{pmatrix} b_{23} \\ b_{33} \end{pmatrix},$$

we have that B' has rank 2, and hence C has rank 2. Thus $\overline{x}_1, \overline{y}_1, \overline{z}_1$ are regular parameters in $\hat{\mathcal{O}}_{X_1,q}$. We have

$$x = \overline{x}_1^{a_{11}}, y = \overline{x}_1^{a_{21}}(\overline{y}_1 + \overline{\alpha}), z = \overline{x}_1^{a_{31}}(\overline{z}_1 + \overline{\beta}).$$

We have that $u = (x^a y^b)^\ell$. Let

$$t = -\frac{b}{a_{11}a + a_{21}b},$$

and set $\overline{x}_1 = x_1(y_1+\overline{\alpha})^t$. Define $\overline{y}_1 = y_1$, $\tilde{\alpha} = \overline{\alpha}$, $\tilde{\beta} = \overline{\alpha}^{ta_{31}}\overline{\beta}$ and $z_1 = (\overline{y}_1+\overline{\alpha})^{ta_{31}}(z_1+\overline{\beta})-\tilde{\beta}$. Then x_1, y_1, z_1 are permissible parameters at q, with $u = x_1^{(aa_{11}+ba_{21})l}$,

$$x = x_1^{a_{11}}(y_1 + \tilde{\alpha})^{ta_{11}}, y = x_1^{a_{21}}(y_1 + \tilde{\alpha})^{ta_{21}+1}, z = x_1^{a_{31}}(z_1 + \tilde{\beta}).$$

Thus we have shown that there exist (formal) permissible parameters x_1, y_1, z_1 at q such that

$$x = x_1^{e_1}(y_1 + \tilde{\alpha})^{\lambda_1}, y = x_1^{e_2}(y_1 + \tilde{\alpha})^{\lambda_2}, z = x_1^{e_3}(z_1 + \tilde{\beta})$$

where $e_1, e_2, e_3 \in \mathbb{Z}_+$, $\tilde{\alpha}, \tilde{\beta} \in \mathfrak{k}$ are nonzero, $\lambda_1, \lambda_2 \in \mathbb{Q}$ are both nonzero, and $u = x_1^{b_1 l}$, where $b_1 = ae_1 + be_2$, $a\lambda_1 + b\lambda_2 = 0$. We then have an expression

$$v = P(x_1^{ae_1+be_2}) + x_1^{ce_1+de_2}G,$$

where

$$\begin{aligned} G \;=\; & (y_1 + \tilde{\alpha})^{c\lambda_1+d\lambda_2}[\tau_0 x_1^{e_3 m}(z_1 + \tilde{\beta})^m \\ & +\tau_2 x_1^{r_2e_1+s_2e_2+(m-2)e_3}(y_1 + \tilde{\alpha})^{r_2\lambda_1+s_2\lambda_2}(z_1 + \tilde{\beta})^{m-2} + \cdots \\ & +\tau_{m-1} x_1^{r_{m-1}e_1+s_{m-1}e_2+e_3}(y_1 + \tilde{\alpha})^{r_{m-1}\lambda_1+s_{m-1}\lambda_2}(z_1 + \tilde{\beta}) \\ & +\tau_m x_1^{r_me_1+s_me_2} y_1^{r_m\lambda_1+s_m\lambda_2}]. \end{aligned}$$

Let $\tau' = \tau_0(0,0,0)$. Let x_1^s be a generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let $G' = \frac{F}{x_1^s}$.

If z^m is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, then G' has an expression

$$G' = \tau'\tilde{\alpha}^{\varphi}(z_1 + \tilde{\beta})^m + g_2(z_1 + \tilde{\beta})^{m-2} + \cdots + g_{m-1}(z + \tilde{\beta}) + g_m + x_1\Omega_1 + y_1\Omega_2$$

for some $g_i \in \mathfrak{k}$ and $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U_1,q}$, where $\varphi = c\lambda_1 + d\lambda_2$. Setting $F' = G' - G'(x_1, 0, 0)$, and $P'(x_1) = P(x_1^{ae_1+be_2}) + x_1^{ce_1+de_2+s}G'(x_1, 0, 0)$, we have that

$$u = x_1^{b_1 l}, v = P'(x_1) + x_1^{ce_1+de_2+s}F'$$

has the form (1) and $\sigma_D(q) \leq \text{ord } F'(0,0,z_1) - 1 \leq m - 2 < m - 1 = \sigma_D(p)$ since $0 \neq \tilde{\beta}$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, but there exists some i with $2 \le i \le m-1$ such that $\tau_i x^{r_i} y^{s_i} z^{m-i}$ is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let h be the smallest i with this property. Then G' has an expression

$$G' = g_h(z_1 + \tilde{\beta})^{m-h} + \cdots + g_{m-1}(z_1 + \tilde{\beta}) + g_m + x_1\Omega_1 + y_2\Omega_2$$

for some $g_i \in \mathfrak{k}$ with $g_h \ne 0$ As in the previous case, we have

$$\sigma_D(q) \le m - h - 1 < m - 1 = \sigma_D(p).$$

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, and $\tau_i x^{r_i} y^{s_i} z^{m-i}$ is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$ for $2 \le i \le m$. Then $x^{r_s} y^{r_s}$ is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, and G' has an expression

$$G' = \tau'_m (y_1 + \tilde{\alpha})^{\varphi + r_m\lambda_1 + s_m\lambda_2} + x_1\Omega$$

where $\tau'_m = \tau_m(0,0,0)$ for some $\Omega \in \hat{\mathcal{O}}_{U_1,q}$. Suppose, if possible, that $\varphi + r_m\lambda_1 + s_m\lambda_2 = 0$. Since $\varphi + r_m\lambda_1 + s_m\lambda_2 = (c + r_m)\lambda_1 + (d + s_m)\lambda_2$, we then have that the nonzero vector (λ_1, λ_2) satisfies $a\lambda_1 + b\lambda_2 = (c + r_m)\lambda_1 + (d + s_m)\lambda_2 = 0$. Thus the determinant $a(d + s_m) - b(c + r_m) = 0$, a contradiction to our assumption that F satisfies (2).

Now since $\varphi + r_m\lambda_1 + s_m\lambda_2 \ne 0$ and $\tilde{\alpha} \ne 0$, we have $1 = \text{ord } G'(0, y_1, 0) < m$, so that $\sigma_D(q) = 0 < m - 1 = \sigma_D(p)$.

Suppose that $q \in \psi^{-1}(p)$ is a 2-point of $\psi^{-1}(\overline{D})$. Then there exist (formal) permissible parameters $\hat{x}_1, \hat{y}_1, \hat{z}_1$ at q such that

$$x = \hat{x}_1^{e_{11}} \hat{y}_1^{e_{12}} (\hat{z}_1 + \hat{\alpha})^{e_{13}}, y = \hat{x}_1^{e_{21}} \hat{y}_1^{e_{22}} (\hat{z}_1 + \hat{\alpha})^{e_{23}}, z = \hat{x}_1^{e_{31}} \hat{y}_1^{e_{32}} (\hat{z}_1 + \hat{\alpha})^{e_{33}} \tag{44}$$

where $e_{ij} \in \mathbb{N}$, with $\text{Det}(e_{ij}) = \pm 1$, and $\hat{\alpha} \in \mathfrak{k}$ is nonzero. We further have

$$e_{11} + e_{12} > 0, e_{21} + e_{22} > 0 \text{ and } e_{31} + e_{32} > 0.$$

First suppose that $e_{11}e_{22} - e_{12}e_{21} \ne 0$. Then q is a 2-point of D_{U_1}.

There exist $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that upon setting

$$\hat{x}_1 = x_1(z_1 + \hat{\alpha})^{\lambda_1} \text{ and } \hat{y}_1 = y_1(z_1 + \hat{\alpha})^{\lambda_2},$$

we have

$$x = x_1^{e_{11}} y_1^{e_{12}}, y = x_1^{e_{21}} y_1^{e_{22}}, z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \hat{\alpha})^r,$$

where

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}.$$

By Cramer's rule,

$$r = \pm \frac{1}{e_{11}e_{22} - e_{12}e_{21}} \ne 0.$$

Now set $z_1 = (z_1 + \hat{\alpha})^r - \hat{\alpha}^r$ and $\alpha = \hat{\alpha}^r$ to obtain permissible parameters x_1, y_1, z_1 at q with

$$x = x_1^{e_{11}} y_1^{e_{12}}, y = x_1^{e_{21}} y_1^{e_{22}}, z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \alpha).$$

We have an expression

$$u = ((x_1^{e_{11}} y_1^{e_{12}})^a (x_1^{e_{21}} y_1^{e_{22}})^b)^\ell = (x_1^{t_1} y_1^{t_2})^{\ell_1}$$

where $t_1, t_2, \ell_1 \in \mathbb{Z}_+$ and $\gcd(t_1, t_2) = 1$.

We then have an expression

$$v = P((x_1^{t_1} y_1^{t_2})^{\frac{\ell_1}{\ell}}) + x_1^{ce_{11} + de_{21}} y_1^{ce_{12} + de_{22}} G,$$

where

$$\begin{aligned} G &= [\tau_0 x_1^{me_{31}} y_1^{me_{32}}(z_1+\alpha)^m + \tau_2 x_1^{r_2e_{11}+s_2e_{21}+(m-2)e_{31}} y_1^{r_2e_{12}+s_2e_{22}+(m-2)e_{32}}(z_1+\alpha)^{m-2}+\cdots \\ &\quad +\tau_{m-1}x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}} y_1^{r_{m-1}e_{12}+s_{m-1}e_{22}+e_{32}}(z_1+\beta)+\tau_m x_1^{r_me_{11}+s_me_{21}} y_1^{r_me_{12}+s_me_{22}}]. \end{aligned}$$

Let $\tau' = \overline{\tau}_0(0,0,0)$. Let $x_1^s y_1^t$ be a generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let $G' = \frac{G}{x_1^s y_1^t}$.

If z^m is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, then G' has an expression

$$G' = \tau'(z_1+\alpha)^m + g_2(z_1+\alpha)^{m-2}+\cdots+g_{m-1}(z-\alpha)+g_m+x_1\Omega_1+y_1\Omega_2$$

for some $g_i \in \mathfrak{k}$ and $\Omega_1,\Omega_2 \in \hat{\mathcal{O}}_{U_1,q}$. Let

$$\overline{P}(x_1^{t_1}y_1^{t_2}) = \sum_{t_2i-t_1j=0} \frac{1}{i!j!}\frac{\partial(x_1^{ce_{11}+de_{21}}y_1^{ce_{12}+de_{22}}G)}{\partial x_1^i\partial y_1^j}(0,0,0)x_1^iy_1^j \tag{45}$$

and $F' = G' - \frac{\overline{P}(x_1^{t_1}y_1^{t_2})}{x_1^{ce_{11}+de_{21}+s}y_1^{ce_{12}+de_{22}+t}}$. Set $P'(x_1^{t_1}y_1^{t_2}) = P((x_1^{t_1}y_1^{t_2})^{\frac{\ell_1}{\ell}}) + \overline{P}(x_1^{t_1}y_1^{t_2})$. We have that

$$u = (x_1^{t_1}y_1^{t_2})^{\ell_1}, v = P'(x_1^{t_1}y_1^{t_2}) + x_1^{ce_{11}+de_{21}+s}y_1^{ce_{12}+de_{22}+t}F'$$

has the form (2), and $\sigma_D(q) = \text{ord } F'(0,0,z_1)-1 \le m-2 < m-1 = \sigma_D(p)$ since $0\ne\alpha$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, but there exists some i with $2\le i\le m-1$ such that $\tau_i x^{r_i}y^{s_i}z^{m-i}$ is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let h be the smallest i with this property. Then G' has an expression

$$G' = g_h(z_1+\beta)^{m-h}+\cdots+g_m+x_1\Omega_1+y_2\Omega_2$$

for some $g_i\in\mathfrak{k}$ with $g_h\ne 0$ As in the previous case, we have $\sigma_D(q)\le m-h-1 < m-1 = \sigma_D(p)$.

Suppose that z^m is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, and $\tau_i x^{r_i}y^{s_i}z^{m-i}$ is not a local generator of $I\hat{\mathcal{O}}_{U_1,q}$ for $2\le i\le m-1$. Then $x^{r_m}y^{r_m}$ is a local generator of $I\hat{\mathcal{O}}_{U_1,q}$, and then G' has an expression

$$G' = 1 + x_1\Omega_1 + y_1\Omega_2$$

for some $\Omega_1,\Omega_2\in\hat{\mathcal{O}}_{U_1,q}$.

We now claim that after replacing G' with $F' = G' - \frac{\overline{P}(x_1^{t_1}y_1^{t_2})}{x_1^{ce_{11}+de_{21}+s}y_1^{ce_{12}+de_{22}+t}}$, where $\overline{P}$ is defined by (45), we have that $F'(0,0,0)\ne 0$. If this were not the case, we would have

$$\begin{aligned} 0 &= \text{Det}\begin{pmatrix} (c+r_m)e_{11}+(d+s_m)e_{21} & (c+r_m)e_{12}+(d+s_m)e_{22} \\ ae_{11}+be_{21} & ae_{12}+be_{22}\end{pmatrix} \\ &= \text{Det}\begin{pmatrix} c+r_m & d+s_m \\ a & b\end{pmatrix}\text{Det}\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22}\end{pmatrix}. \end{aligned}$$

Since $e_{11}e_{22}-e_{21}e_{12}\ne 0$ (by our assumption), we get

$$0 = \text{Det}\begin{pmatrix} c+r_m & d+s_m \\ a & b\end{pmatrix}$$

which is a contradiction to our assumption that F satisfies (2). Since $F'(0,0,0)\ne 0$, we have that $\sigma_D(q) = 0 < m-1 = \sigma_D(p)$.

Now suppose that q is a 2-point of $\psi^{-1}(\overline{D})$ with $e_{11}e_{22}-e_{21}e_{12}=0$ in (44).

We make a substitution

$$\hat{x}_1 = x_1(z_1+\alpha)^{\varphi_1}, \hat{y}_1 = y_1(z_1+\alpha)^{\varphi_2}, \hat{z}_1 = z_1$$

where $\alpha = \hat{\alpha}$ and $\varphi_1, \varphi_2 \in \mathbb{Q}$ satisfy

$$\begin{array}{rcl} 0 & = & a(\varphi_1 e_{11} + \varphi_2 e_{12} + e_{13}) + b(\varphi_1 e_{21} + \varphi_2 e_{22} + e_{23}) \\ & = & \varphi_1(ae_{11} + be_{21}) + \varphi_2(ae_{12} + be_{22}) + ae_{13} + be_{23}. \end{array}$$

We have $ae_{11} + be_{21} > 0$ and $ae_{12} + be_{22} > 0$ since $a, b > 0$ and by the condition satisfied by the e_{ij} stated after (44).

Let

$$\lambda_1 = \varphi_1 e_{11} + \varphi_2 e_{12} + e_{13}, \lambda_2 = \varphi_1 e_{21} + \varphi_2 e_{22} + e_{23}, \lambda_3 = \varphi_1 e_{31} + \varphi_2 e_{32} + e_{33}.$$

Then x_1, y_1, z_1 are permissible parameters at q such that

$$x = x_1^{e_{11}} y_1^{e_{12}} (z_1 + \alpha)^{\lambda_1}, y = x_1^{e_{21}} y_1^{e_{22}} (z_1 + \alpha)^{\lambda_2}, z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \alpha)^{\lambda_3} \tag{46}$$

with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}$, and $a\lambda_1 + b\lambda_2 = 0$.

Now suppose that $e_{11} > 0$ and $e_{12} > 0$, which is the case where q is a 2-point of D_{U_1}. Write

$$u = ((x_1^{e_{11}} y_1^{e_{12}})^a (x_1^{e_{21}} y_1^{e_{22}})^b)^\ell = (x_1^{t_1} y_1^{t_2})^{\ell_1}$$

where $t_1, t_2, \ell_1 \in \mathbb{Z}_+$ and $\gcd(t_1, t_2) = 1$.

We then have an expression

$$v = P((x_1^{t_1} y_1^{t_2})^{\frac{\ell_1}{\ell}}) + x_1^{ce_{11}+de_{21}} y_1^{ce_{12}+de_{22}} G,$$

where

$$\begin{array}{rcl} G & = & (z_1+\alpha)^{c\lambda_1+d\lambda_2}[\tau_0 x_1^{me_{31}} y_1^{me_{32}} (z_1+\alpha)^{m\lambda_3} \\ & & +\tau_2 x_1^{r_2e_{11}+s_2e_{21}+(m-2)e_{31}} y_1^{r_2e_{12}+s_2e_{22}+(m-2)e_{32}} (z_1+\alpha)^{r_2\lambda_1+s_2\lambda_2+(m-2)\lambda_3} + \cdots \\ & & +\tau_{m-1} x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}} y_1^{r_{m-1}e_{12}+s_{m-1}e_{22}+e_{32}} (z_1+\alpha)^{\lambda_1 r_{m-1}+\lambda_2 s_{m-1}+\lambda_3} \\ & & +\tau_m x_1^{r_m e_{11}+s_m e_{21}} y_1^{r_m e_{12}+s_m e_{22}} (z_1+\alpha)^{r_m\lambda_1+s_m\lambda_2}]. \end{array}$$

Let $x_1^s y_1^t$ be a generator of $I\hat{\mathcal{O}}_{U_1,q}$. Let $G' = \frac{F}{x_1^s y_1^t}$.

We will now establish that, with our assumptions, there is a unique element of the set S consisting of z^m, and

$$\{x^{r_i} y^{s_i} z^{m-i} \mid 2 \le i \le m \text{ and } \tau_i \ne 0\}$$

which is a generator of $I\hat{\mathcal{O}}_{U_1,q}$; that is, is equal to $x_1^s y_1^t$ times a unit in $\hat{\mathcal{O}}_{U_1,q}$. Let $r_0 = 0$ and $s_0 = 0$. Suppose that $x^{r_i} y^{r_i} z^{m-i}$ (with $0 \le i \le m$) is a generator of $I\hat{\mathcal{O}}_{U_1,q}$. We have $x^{r_i} y^{s_i} z^{m-i} = x_1^s y_1^t (z_1 + \alpha)^{\gamma_i}$ where

$$\begin{array}{c} r_i e_{11} + s_i e_{21} + (m-i) e_{31} = s \\ r_i e_{12} + s_i e_{22} + (m-i) e_{32} = t \\ r_i \lambda_1 + s_i \lambda_2 + (m-i)\lambda_3 = \gamma_i. \end{array}$$

Let

$$A = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}. \tag{47}$$

We have

$$A \begin{pmatrix} r_i \\ s_i \\ m-i \end{pmatrix} = \begin{pmatrix} s \\ t \\ \gamma_i \end{pmatrix}. \tag{48}$$

Let $\omega = \mathrm{Det}(A)$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_1 & \varphi_2 & 1 \end{pmatrix} \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}$$

implies $\omega = \mathrm{Det}(A) = \pm 1$.

By Cramer's rule, we have

$$\begin{aligned} \omega(m-i) &= \mathrm{Det}\begin{pmatrix} e_{11} & e_{21} & s \\ e_{12} & e_{22} & t \\ \lambda_1 & \lambda_2 & \gamma_i \end{pmatrix} \\ &= s\mathrm{Det}\begin{pmatrix} e_{12} & e_{22} \\ \lambda_1 & \lambda_2 \end{pmatrix} - t\mathrm{Det}\begin{pmatrix} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{pmatrix} + \gamma_i \mathrm{Det}\begin{pmatrix} e_{11} & e_{21} \\ e_{12} & e_{22} \end{pmatrix}. \end{aligned}$$

Since $e_{11}e_{21} - e_{12}e_{22} = 0$ by assumption, we have that

$$i = m - \frac{1}{\omega}\left(s\mathrm{Det}\begin{pmatrix} e_{12} & e_{22} \\ \lambda_1 & \lambda_2 \end{pmatrix} - t\mathrm{Det}\begin{pmatrix} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{pmatrix}\right).$$

In particular, there is a unique element $x^{r_i}y^{r_i}z^{m-i} \in S$ which is a generator of $I\hat{\mathcal{O}}_{U_1,q}$. We have $x^{r_i}y^{s_i}z^{m-i} = x_1^s t_1^t (z_1+\alpha)^{\gamma_i}$.

We thus have that $G = x_1^s y_1^t [g(z_1+\alpha)^{\gamma_i + c\lambda_1 + d\lambda_2} + x_1\Omega_1 + y_1\Omega_2]$ for some $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U_1,q}$ and $0 \neq g \in \mathfrak{k}$.

We now establish that we cannot have that $\gamma_i + c\lambda_1 + d\lambda_2 = 0$ and $x_1^{ce_{11}+de_{21}+s} y_1^{ce_{12}+de_{22}+t}$ is a power of $x_1^{t_1}y_1^{t_2}$. We will suppose that both of these conditions do hold, and derive a contradiction. Now we know that $x^a y^b = x_1^{ae_{11}+be_{21}} y_1^{ae_{12}+be_{22}}$ is a power of $x_1^{t_1}y_1^{t_2}$. By (47), (48) and our assumptions, we have that

$$A\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

and

$$A\begin{pmatrix} c + r_i \\ d + s_i \\ m - i \end{pmatrix}$$

are rational multiples of

$$\begin{pmatrix} t_1 \\ t_2 \\ 0 \end{pmatrix}.$$

Since $\omega = \mathrm{Det}(A) \neq 0$, we have that $(c+r_i, d+s_i, m-i)$ is a rational multiple of $(a,b,0)$. Thus $x^c y^d x^{r_i} y^{s_i} z^{m-i}$ is a power of $x^a y^b$, a contradiction to our assumption that F satisfies (2).

Let

$$\overline{P}(x_1^{t_1}y_1^{t_2}) = \sum_{t_2 i - t_1 j = 0} \frac{1}{i!j!} \frac{\partial (x_1^{ce_{11}+de_{21}} y_1^{ce_{12}+de_{22}} G)}{\partial x_1^i \partial y_1^j}(0,0,0) x_1^i y_1^j,$$

and $F' = G' - \frac{\overline{P}(x_1^{t_1}y_1^{t_2})}{x_1^{ce_{11}+de_{21}+s} y_1^{ce_{12}+de_{22}+t}}$. Set

$P'(x_1^{t_1}y_1^{t_2}) = P((x_1^{t_1}y_1^{t_2})^{\frac{\ell_1}{\ell}}) + \overline{P}(x_1^{t_1}y_1^{t_2})$. We have that

$$u = (x_1^{t_1}y_1^{t_2})^{\ell_1}, v = P'(x_1^{t_1}y_1^{t_2}) + x_1^{ee_{11}+fe_{21}} y_1^{ce_{21}+de_{22}} F'$$

has the form (2) and $\sigma_D(q) = 0 \le m-2 = \sigma_D(p)$.

Now suppose that $q \in \psi^{-1}(p)$ is a 2-point of $\psi^{-1}(\overline{D})$, $e_{11}e_{22} - e_{12}e_{21} = 0$ in (44), and $e_{11} = 0$ or $e_{12} = 0$. Without loss of generality, we may assume that $e_{12} = 0$. q is a 1-point of D_{U_1}, and we have permissible parameters (46) at q. Since $\mathrm{Det}(e_{ij}) = \pm 1$, we have that $e_{32} = 1$, and $e_{11}e_{23} - e_{21}e_{13} = \pm 1$. Replacing y_1 with $y_1(z_1+\alpha)^{\lambda_3}$ in (46), we find permissible parameters x_1, y_1, z_1 at q such that

$$x = x_1^{e_{11}}(z_1+\alpha)^{\lambda_1},\ y = x_1^{e_{21}}(z_1+\alpha)^{\lambda_2},\ z = x_1^{e_{31}}y_1, \tag{49}$$

with $e_{11}, e_{21} > 0$ and $a\lambda_1 + b\lambda_2 = 0$. We have

$$\begin{array}{rcl} u & = & x_1^{(ae_{11}+be_{21})l} = x_1^{l_1} \\ v & = & P(x_1^{ae_{11}+be_{21}}) + x_1^{ce_{11}+de_{21}}G \end{array}$$

where

$$\begin{array}{rcl} G & = & (z_1+\alpha)^{c\lambda_1+d\lambda_2}[\tau_0 x_1^{me_{31}}y_1^m + \tau_2 x_1^{r_2e_{11}+s_2e_{21}+(m-2)e_{31}}y_1^{m-2}(z_1+\alpha)^{r_2\lambda_1+s_2\lambda_2} + \cdots \\ & & +\tau_{m-1}x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}}y_1(z_1+\alpha)^{r_{m-1}\lambda_1+s_{m-1}\lambda_2} \\ & & +\tau_m x_1^{r_me_{11}+s_me_{21}}(z_1+\alpha)^{r_m\lambda_1+s_m\lambda_2}]. \end{array}$$

Since $I\hat{\mathcal{O}_{U_1,q}}$ is principal and τ_m or $\tau_{m-1} \ne 0$, we have that $x_1^{r_me_{11}+s_me_{21}}$ is a generator of $I\hat{\mathcal{O}}_{U_1,q}$ if $\tau_m \ne 0$, and $x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}}y_1$ is a generator of $I\hat{\mathcal{O}}_{U_1,q}$ if $\tau_m = 0$ and $\tau_{m-1} \ne 0$.

First suppose that $\tau_m \ne 0$ so that

$$G = x_1^{r_me_{11}+s_me_{21}}[g_m(z_1+\alpha)^{(c+r_m)\lambda_1+(d+s_m)\lambda_2} + x_1\Omega + y_1\Omega_2]$$

with $0 \ne g_m \in \mathfrak{k}$, $\Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U_1,q}$. Since λ_1, λ_2 are not both zero, $a\lambda_1 + b\lambda_2 = 0$ and $a(d+s_m) - b(c+r_m) \ne 0$, we have that $(c+r_m)\lambda_1 + (d+s_m)\lambda_2 \ne 0$. Let $\overline{P}(x_1) = G(x_1, 0, 0)$, and $P'(x_1) = P(x_1^{ae_{11}+be_{21}}) + \overline{P}(x_1)$. Let

$$F' = \frac{1}{x_1^{ce_{11}+de_{21}}}(G - \overline{P}(x_1)).$$

Then

$$\begin{array}{rcl} u & = & x_1^{l_1} \\ v & = & P'(x_1) + x_1^{ce_{11}+de_{21}}F' \end{array}$$

is of the form (1) with $\mathrm{ord}\, F'(0, y_1, z_1) = 1$. Thus $\sigma_D(q) = 0 < \sigma_D(p)$.

Now suppose that $\tau_m = 0$, so that

$$G = x_1^{r_{m-1}e_{11}+s_{m-1}e_{21}+e_{31}}[g_{m-1}y_1(z_1+\alpha)^{(c+r_{m-1})\lambda_1+(d+s_{m-1})\lambda_2} + x_1\Omega_1]$$

with $0 \ne g_{m-1} \in \mathfrak{k}$ and $\Omega_1 \in \hat{\mathcal{O}}_{U_1,q}$. Thus $\sigma_D(q) = 0 < \sigma_D(p)$.

The final case is when q is a 3-point for $\psi^{-1}(\overline{D})$, so that q is a 3-point or a 2-point of D_{U_1}. Then we have permissible parameters x_1, y_1, z_1 at q such that

$$x = x_1^{e_{11}}y_1^{e_{12}}z_1^{e_{13}}, y = x_1^{e_{21}}y_1^{e_{22}}z_1^{e_{23}}, z = x_1^{e_{31}}y_1^{e_{32}}z_1^{e_{33}}$$

with $\omega = \mathrm{Det}(e_{ij}) = \pm 1$. Thus there is a unique element of the set S consisting of z^m and

$$\{x^{r_i}y^{s_i}z^{m-i} \mid 2 \le i \le m \text{ and } \overline{\tau}_i \ne 0\}$$

which is a generator $x_1^{s_1}y_1^{s_2}z_1^{s_3}$ of $I\hat{\mathcal{O}}_{U',q}$. Thus $\sigma_D(q) = 0$ if q is a 3-point of D_{U_1}. If q is a 2-point of D_{U_1}, we may assume that $e_{13} = e_{23} = 0$. Then $e_{33} = 1$. Since $\tau_m \ne 0$ or $\tau_{m-1} \ne 0$, we calculate that $\sigma_D(q) = 0$.

$\square$

Suppose that $p \in X$ is a 2-point such that X is 3-prepared at p and $\sigma_D(p) = r > 0$. We can then define a local resolver $(U_p, \overline{D}_p, I_p, \nu_p^1, \nu_p^2)$ as in Theorem 4.3, where ν_p^i are valuations on U_p which dominate the two curves C_1, C_2 which are the intersection of E with D_{U_p} on U_p (where $\overline{D}_p = D_{U_p} + E$), and which have the property that if $\pi : V \to U_p$ is a birational morphism, then the center $C(V, \nu_p^i)$ on V is the unique curve on the strict transform of E on V which dominates C_i. We will think of U_p as a germ, so we will feel free to replace U_p with a smaller neighborhood of p whenever it is convenient.

If $\pi : Y \to X$ is a birational morphism, then the center $C(Y, \nu_p^i)$ on Y is the closed curve which is the center of ν_p^i on Y. We define $\Lambda(Y, \nu_p^i)$ to be the image in Y of $C(Y \times_X U_p, \nu_p^i) \cap \pi^{-1}(p)$. This defines a valuation which is composite with $C(Y, \nu_p^i)$.

We define $W(Y, p)$ to be the clopen locus on Y of the image of points in $\pi^{-1}(U_p) = Y \times_X U_p$ such that $I_p\mathcal{O}_Y \mid \pi^{-1}(U_p)$ is not invertible. Define $\mathrm{Preimage}(Y, Z) = \pi^{-1}(Z)$ for Z a subset of X.

5. Global reduction of σ_D

Lemma 5.1. *Suppose that X is 2-prepared and $p \in X$ is 3-prepared. Suppose that $r = \sigma_D(p) > 0$.*

a) *Suppose that p is a 1-point. Then there exists a unique curve C in $\mathrm{Sing}_1(X)$ containing p. The curve C is contained in $\mathrm{Sing}_r(X)$. If x, y, z are permissible parameters at p giving an expression (14) or (15) at p, then $z = z = 0$ are formal local equations of C at p.*

b) *Suppose that p is a 2-point and C is a curve in $\mathrm{Sing}_r(X)$ containing p. If x, y, z are permissible parameters at p giving an expression (13) at p, then $x = z = 0$ or $y = z = 0$ are formal local equations of C at p.*

Proof. We first prove a). Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf defining the reduced scheme $\mathrm{Sing}_1(X)$. Then $\mathcal{I}_p\hat{\mathcal{O}}_{X,p} = \sqrt{(x, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})} = (x, z)$ is an ideal on U defining $\mathrm{Sing}_1(U)$. Thus the unique curve C in $\mathrm{Sing}_1(X)$ through p has (formal) local equations $x = z = 0$ at p. At points near p on C, a form (14) or (15) continues to hold with $m = r + 1$. Thus the curve is in $\mathrm{Sing}_r(X)$.

We now prove b). Suppose that $C \subset \mathrm{Sing}_r(X)$ is a curve containing p. By Theorem 4.3, there exists a toroidal morphism $\Psi : U_1 \to U$ where U is an étale cover of an affine neighborhood of p, and $\overline{D}$ is the local toroidal structure on U defined (formally at p) by $xyz = 0$, such that all points q of U_1 satisfy $\sigma_D(q) < r$. Hence the strict transform on U_1 of the preimage of C on U must be empty. Since Ψ is toroidal for $\overline{D}$ and X is 3-prepared at p, C must have local equations $x = z = 0$ or $y = z = 0$ at p. $\square$

Definition 5.2. *Suppose that X is 3-prepared. We define a canonical sequence of blow ups over a curve in X.*

1) *Suppose that C is a curve in X such that $t = \sigma_D(q) > 0$ at the generic point q of C, and all points of C are 1-points of D. Then we have that C is nonsingular and $\sigma_D(p) = t$ for all $p \in C$ by Lemma 5.1. By Lemma 5.1 and Theorem 4.1 or 4.2, there exists a unique minimal sequence of permissible blow ups of sections over C, $\pi_1 : X_1 \to X$, such that X_1 is 2-prepared and $\sigma_D(p) < t$ for all $p \in \pi_1^{-1}(C)$. We will call the morphism π_1 the canonical sequence of blow ups over C.*
2) *Suppose that C is a permissible curve in X which contains a 1-point such that $\sigma_D(p) = 0$ for all $p \in C$, and a condition 1, 3 or 5 of Lemma 3.10 holds at all*

$p \in C$. Let $\pi_1 : X_1 \to X$ be the blow up of C. Then by Lemma 3.12, X_1 is 3-prepared and $\sigma_D(p) = 0$ for $p \in \pi_1^{-1}(C)$. We will call the morphism π_1 the canonical blow up of C.

Theorem 5.3. *Suppose that X is 2-prepared. Then there exists a sequence of permissible blowups $\psi : X_1 \to X$ such that X_1 is prepared.*

Proof. By Proposition 3.13, there exists a sequence of permissible blow ups $X^0 \to X$ such that X^0 is 3-prepared. Let $r = \Gamma_D(X^0)$. Since X^0 is prepared if $r = 0$, we may assume that $r > 0$. Let

$$T_0 = \{p \in X^0 \mid X^0 \text{ is a 2-point for } D \text{ with } \sigma_D(p) = r\}.$$

For $p \in T_0$, choose $(U_p, \overline{D}_p, I_p, \nu_p^1, \nu_p^2)$. Let Γ_0 be the union of the set of curves

$$\{C(X^0, \nu_p^j) \mid p \in T_0 \text{ and } \sigma_D(\eta) = r \text{ for } \eta \in C(X^0, \nu_p^j) \text{ the generic point}\}$$

and any remaining curves C in $\mathrm{Sing}_r(X^0)$ (which necessarily contain no 2-points).

By Lemma 5.1, all curves in $\mathrm{Sing}_r(X^0)$ are nonsingular, and if a curve C in $\mathrm{Sing}_r(X^0)$ contains a 2-point $p \in T_0$, then $C = C(X^0, \nu_p^j)$ for some j.

Let $Y_0 \to X^0$ be the product of canonical sequences of blowups over the curves in Γ_0 (which are necessarily the curves in $\mathrm{Sing}_r(X^0)$), so that $Y_0 \setminus \cup_{p\in T_0} W(Y_0, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_0 \setminus \cup_{p\in T_0} W(Y_0, p)$.

Let $Y_{0,1} \to Y_0$ be a torodial morphism for D_{Y_0} so that the components of $D_{Y_{0,1}}$ containing some curve $C(Y_{0,1}, \nu_p^j)$ for $p \in T_0$ are pairwise disjoint, and if $p \in T_0$, then $W(Y_{0,1}, p)$ is contained in $C(Y_{0,1}, \nu_p^1) \cup C(Y_{0,1}, \nu_p^2) \cup \mathrm{Preimage}(Y_{0,1}, p)$.

Let E be a component of $D_{Y_{0,1}}$ which contains $C(Y_{0,1}, \nu_p^j)$ for some $p \in T_0$ and some j. Then there exists $Y_{0,2} \to Y_{0,1}$ which is an isomorphism over $Y_{0,1} \setminus E \cap (\cup_{p\in T_0} W(Y_{0,1}, p))$, is toroidal for $\overline{D}_q$ over $W(Y_{0,1}, q) \cap E$ for $q \in T_0$, is an isomorphism over $C(Y_{0,1}, \nu_q^j) \setminus \mathrm{Preimage}(q)$ for all $q \in T_0$, and so that if $\overline{E}$ is the strict transform of E on $Y_{0,2}$, then for $p \in T_0$, one of the following holds:

$$W(Y_{0,2}, p) \cap \overline{E} = \emptyset \tag{50}$$

or

(51) There exists a unique j such that $W(Y_{0,2}, p) \cap \overline{E} \subset C(Y_{0,2}, \nu_p^j) \subset \overline{E}$, and if $\overline{p}_j = \Lambda(Y_{0,2}, \nu_p^j)$, then $C(Y_{0,2}, \nu_p^j)$ is smooth at $\overline{p}_j$, and either $\overline{p}_j$ is an isolated point in $\mathrm{Sing}_1(Y_{0,2})$ or $C(Y_{0,2}, \nu_p^j)$ is the only curve in $\mathrm{Sing}_1(Y_{0,2})$ which is contained in $\overline{E}$ and contains $\overline{p}_j$, and $\overline{p}_j \in C(Y_{0,2}, \nu_{p'}^k)$ for some $p' \in T_0$ implies $C(Y_{0,2}, \nu_{p'}^k) = C(Y_{0,2}, \nu_p^j)$.

We further have that $Y_{0,2} \setminus \cup_{p\in T_0} W(Y_{0,2}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{0,2} \setminus \cup_{p\in T_0} W(Y_{0,2}, p)$.

Now repeat this procedure for other components of $D_{Y_{0,2}}$ which contain a curve $C(Y_{0,2}, \nu_p^j)$ for some j to construct $Y_{0,3} \to Y_{0,2}$ so that condition (50) or (51) hold for all components E of $D_{Y_{0,3}}$ containing a curve $C(Y_{0,3}, \nu_p^j)$. We have that $Y_{0,3} \setminus \cup_{p\in T_0} W(Y_{0,3}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{0,3} \setminus \cup_{p\in T_0} W(Y_{0,3}, p)$.

Now, by Lemma 3.4, let $Y_{0,4} \to Y_{0,3}$ be a sequence of blow ups of 3-points for D and 2-curves of D on the strict transform of components E of D which contain $C(Y_{0,3}, \nu_p^j)$ for some $p \in T_0$, so that if E is a component of $D_{Y_{0,4}}$ which contains a curve $C(Y_{0,4}, \nu_p^j)$, then $Y_{0,4}$ is 3-prepared at all 2-points and 3-points of E. We have that $Y_{0,4} \setminus \cup_{p \in T_0} W(Y_{0,4}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{0,4} \setminus \cup_{p \in T_0} W(Y_{0,4}, p)$. We further have that for all $p \in T_0$, (50) or (51) holds on E.

Now let E be a component of $D_{Y_{0,4}}$ which contains a curve $C(Y_{0,4}, \nu_p^j)$. Since one of the conditions (50) or (51) hold for all $p \in T_0$ on E, we may apply Proposition 3.14 to E and the finitely many points

$$A = \{q \in E \mid Y_{0,4} \text{ is not 3-prepared at } q\},$$

which are necessarily 1-points for D, being sure that none of the finitely many 2-points for D

$$B = \{\Lambda(Y_{0,4}, \nu_p^j) \mid p \in T_0\}$$

are in the image of the general curves blown up, to construct a sequence of permissible blow ups $Y_{0,5} \to Y_{0,4}$ so that $Y_{0,5} \to Y_{0,4}$ is an isomorphism in a neighborhood of $\cup_{p \in T_0} W(Y_{0,4}, p)$ and over $Y_{0,4} \setminus E$, and $Y_{0,5}$ is 3-prepared over $E \setminus \cup_{p \in T_0} \Lambda(Y_{0,4}, \nu_p^j)$. We have that $Y_{0,5} \setminus \cup_{p \in T_0} W(Y_{0,5}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{0,5} \setminus \cup_{p \in T_0} W(Y_{0,5}, p)$. We further have that for all $p \in T_0$, (50) or (51) hold on the strict transform $\overline{E}$ of E on $Y_{0,5}$.

Now repeat this procedure for other components of $D_{Y_{0,5}}$ which contain a curve $C(Y_{0,5}, \nu_p^j)$ for some j to construct $X_1 \to Y_{0,5}$ so that X_1 is 3-prepared over $E \setminus \cup_{p \in T_0} \Lambda(Y_{0,5}, \nu_p^j)$ for all components E of $D_{Y_{0,5}}$ which contain a curve $C(Y_{0,5}, \nu_p^j)$ for some $p \in T_0$. We then have that the following holds.

1.1) $X_1 \to X^0$ is the canonical sequence of blow ups above a general point η of a curve in Γ_0 (so that $\sigma_D(\eta) = r$).
1.2) $X_1 \to X^0$ is toroidal for $\overline{D}_p$ in a neighborhood of $W(X_1, p)$, for $p \in T_0$.
1.3) $X_1 \setminus \cup_{p \in T_0} W(X_1, p)$ is 2-prepared and $\sigma_D(q) < r$ for $q \in X_1 \setminus \cup_{p \in T_0} W(X_1, p)$,
1.4) If $p \in T_0$ then $\sigma_D(q) \le r - 1$ and X_1 is 3-prepared at q for

$$q \in C(X_1, \nu_p^j) \setminus \cup_{p' \in T_0 | C(X_i, \nu_p^j) = C(X_i, \nu_{p'}^k) \text{ for some } k} \text{Preimage}(X_1, p').$$

1.5) Let

$$T_1 = \begin{cases} \text{2-points } q \text{ for } D \text{ of} \\ C(X_1, \nu_p^j) \setminus \cup_{p' \in T_0 | C(X_1, \nu_p^j) = C(X_1, \nu_{p'}^k) \text{ for some } k} \text{Preimage}(X_1, p') \\ \text{such that } \sigma_D(q) > 0 \text{ and such that } p \in T_0 \text{ with} \\ \sigma_D(\eta) = r - 1 \text{ for } \eta \in C(X_1, \nu_p^j) \text{ the generic point.} \end{cases}$$

X_1 is 3-prepared at $p \in T_1$. For $q \in T_1$, choose $(U_q, \overline{D}_q, I_q, \nu_q^1, \nu_q^2)$. We have $0 < \sigma_D(q) \le r - 1$ for $q \in T_1$.
1.6) Suppose that $p \in T_0$ and $C(X_1, \nu_p^j)$ is such that $\sigma_D(\eta) = r - 1$ for $\eta \in C(X_1, \nu_p^j)$ the generic point. Then $\sigma_D(q) = r - 1$ for $q \in C(X_1, \nu_p^j) \setminus \cup_{p' \in T_0 \cup T_1} W(X_1, p')$. If $q \in T_0 \cup T_1$ and $W(X_1, q) \cap C(X_1, \nu_p^j) \neq \emptyset$, then $C(X_1, \nu_p^j) = C(X_1, \nu_q^i)$ for some i. (This follows from Lemma 5.1 since $\sigma_D(q) \le r - 1$ for $q \in T_1$.)

Now for $m \geq r$, we inductively construct
(52)
$$X_{m,r-1} \to \cdots \to X_{m,0}, \to \cdots \to X_{r+1,r-1} \to \cdots \to X_{r+1,0} \to$$
$$X_{r,r-1} \to X_{r,r-2} \to \cdots \to X_{r,0} \to X_{r-1,r-2} \to \cdots \to X_{3,0} \to X_{2,1} \to X_{2,0} \to X_{1,0} = X_1 \to X^0$$
so that

2.1) $X_{1,0} = X_1 \to X^0$ is the canonical sequence of blow ups above a general point η of a curve in Γ_0 (so that $\sigma_D(\eta) = r$), and for $i > 0$,
$$X_{i+1,0} \to X_{i,\min\{i-1,r-1\}}$$
is the canonical sequence of blowups above a general point η of a curve $C(X_{i,\min\{i-1,r-1\}}, \nu_p^j)$ with $p \in T_0$ and such that $\sigma_D(\eta) = \max\{0, r-i\}$,

and the following properties hold on $X_{i,l}$.

2.2) $X_{i,l} \to X_{j,k}$ is toroidal for $\overline{D}_p$ in a neighborhood of $W(X_{i,l}, p)$, for $p \in T_{j,k}$ with $T_{j,k} = T_0$, or $1 \leq j \leq i-1$ and $0 \leq k \leq \min\{j-1, r-1\}$, or $j = i$ and $0 \leq k \leq l-1$.

2.3) $X_{i,l} \setminus \cup_{p \in T_0 \cup \left(\cup_{j=1}^{i-1} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup \left(\cup_{n=0}^{l-1} T_{i,n}\right)} W(X_{i,l}, p)$ is 2-prepared and $\sigma_D(q) < r$ for $q \in X_{i,l} \setminus \cup_{p \in T_0 \cup \left(\cup_{j=1}^{i-1} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup \left(\cup_{n=0}^{l-1} T_{i,n}\right)} W(X_{i,l}, p)$.

2.4) If $p \in T_0$ then $\sigma_D(\eta) \leq \max\{0, r-i\}$ for $\eta \in C(X_{i,l}, \nu_p^j)$ the generic point, and $X_{i,l}$ is 3-prepared at q for
$$q \in C(X_{i,l}, \nu_p^j) \setminus \cup_{p' \in \Omega} \mathrm{Preimage}(X_{i,l}, p'),$$
Where
$$\Omega = \{p' \in T_0 \cup \left(\cup_{j=1}^{i-1} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup \left(\cup_{n=0}^{l-1} T_{i,n}\right) \mid C(X_{i,l}, \nu_p^j) = C(X_{i,l}, \nu_{p'}^k) \text{ for some } k\}.$$

2.5) We have the set
$$T_{i,l} = \left\{ \begin{array}{l} \text{2-points } q \text{ for } D \text{ of } C(X_{i,l}, \nu_p^j) \setminus \cup_{p' \in \Omega} \mathrm{Preimage}(X_{i,l}, p') \\ \text{where } \Omega = \\ \{p' \in T_0 \cup \left(\cup_{j=1}^{i-1} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup \left(\cup_{n=0}^{l-1} T_{i,n}\right) \mid C(X_{i,l}, \nu_p^j) = C(X_{i,l}, \nu_{p'}^k) \text{ for some } k\} \\ \text{such that } \sigma_D(q) > 0 \text{ and such that } p \in T_0 \text{ with} \\ \sigma_D(\eta) = \max\{0, r-i\} \text{ for } \eta \in C(X_{i,l}, \nu_p^j) \text{ the generic point.} \end{array} \right\}$$
$X_{i,l}$ is 3-prepared at $p \in T_{i,l}$. We have local resolvers $(U_p, \overline{D}_p, I_p, \nu_p^1, \nu_p^2)$ at $p \in T_{i,l}$. We have $\max\{1, r-i\} \leq \sigma_D(q) \leq r-l-1$ for $q \in T_{i,l}$.

2.6) Suppose that $p \in T_0$ and $C(X_{i,l}, \nu_p^j)$ is such that $\sigma_D(\eta) = \max\{0, r-i\}$ for $\eta \in C(X_{i,l}, \nu_p^j)$ the generic point. Then $\sigma_D(q) = \max\{0, r-i\}$ for
$$q \in C(X_{i,l}, \nu_p^j) \setminus \cup_{p' \in T_0 \cup \left(\cup_{j=1}^{i-1} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup \left(\cup_{n=0}^{l} T_{i,n}\right)} W(X_{i,l}, p').$$
Further,
   a) If $q \in T_0 \cup \left(\cup_{j=1}^{i-1} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup \left(\cup_{n=0}^{l-1} T_{i,n}\right)$ and $W(X_{i,l}, q) \cap C(X_{i,l}, \nu_p^j) \neq \emptyset$, then $C(X_{i,l}, \nu_p^j) = C(X_{i,l}, \nu_q^k)$ for some k.
   b) If $q \in T_{i,l}$ and $q \in C(X_{i,l}, \nu_p^j)$, then either $C(X_{i,l}, \nu_p^j) = C(X_{i,l}, \nu_q^k)$ for some k or $\max\{0, r-i\} < \sigma_D(q) \leq r-l-1$.

Note that the condition "$\sigma_D(q) > 0$" in the definition of $T_{i,l}$ is automatically satisfied if $i < r$. If $l = \min\{i-1, r-1\}$, condition 2.6) becomes "Suppose that $p \in T_0$ and $C(X_{i,l}, \nu_p^j)$ is such that $\sigma_D(\eta) = \max\{0, r-i\}$ for $\eta \in C(X_{i,l}, \nu_p^j)$ the generic point. Then if $q \in T_0 \cup \left(\cup_{j=1}^{i-1} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup \left(\cup_{n=0}^{l} T_{i,n}\right)$ and $W(X_{i,l}, q) \cap C(X_{i,l}, \nu_p^j) \neq \emptyset$, then $C(X_{i,l}, \nu_p^j) = C(X_{i,l}, \nu_q^k)$ for some k".

We now prove the above inductive construction of (52). Suppose that we have made the construction out to $X_{i,l}$.

Case 1. Suppose that $l = \min\{i-1, r-1\}$. We will construct $X_{i+1,0} \to X_{i,\min\{i-1,r-1\}}$.

First suppose that $r > i$. Let $Y_i \to X_{i,i-1}$ be the product of the canonical sequences of blow ups above all curves $C(X_{i,i-1}, \nu_p^j)$ for $p \in T_0$ such that $\sigma_D(\eta) = r - i$ at a generic point $\eta \in C(X_{i,l}, \nu_p^j)$. This is a permissible sequence of blow ups by the comment following 2.6) above. We have that $Y_i \setminus \cup_{p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(Y_i, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_i \setminus \cup_{p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(Y_i, p)$. Further, $Y_i \to X_{i,i-1}$ is toroidal for $\overline{D}_p$ in a neighborhood of $W(Y_i, p)$ for $p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)$.

Now suppose that $r \leq i$. On $X_{i,r-1}$, we have that $\sigma_D(q) = 0$ for $p \in T_0$ and $q \in C(X_{i,r-1}, \nu_p^j) \setminus \cup_{p' \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(X_{i,r-1}, p')$. By Lemmas 3.9, 3.10, 3.11 and 3.12, there exists a sequence $Y_i \to X_{i,r-1}$ of blow ups of prepared points on the strict transform of curves $C(X_{i,r-1}, \nu_p^j)$ with $p \in T_0$, followed by the blow ups of the strict transforms of these $C(X_{i,r-1}, \nu_p^j)$, so that for $p \in T_0$, $\sigma_D(q) = 0$ at a point q of $C(Y_i, \nu_p^j)$, $Y_i \setminus_{p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(Y_i, p)$ is 2-prepared and $\sigma_D(q) < r$ for

$$q \in Y_i \setminus \cup_{p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(Y_i, p).$$

Further, $Y_i \to X_{i,r-1}$ is toroidal for $\overline{D}_p$ in a neighborhood of $W(Y_i, p)$ for $p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)$.

From now on, we consider both cases $r > i$ and $r \leq i$ simultaneously. Let $Y_{i,1} \to Y_i$ be a torodial morphism for D so that the components of D containing some curve $C(Y_{i,1}, \nu_p^j)$ for $p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)$ are pairwise disjoint, and if

$$p \in \cup_{p' \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(Y_{i,1}, p')$$

then $W(Y_{i,1}, p)$ is contained in $C(Y_{i,1}, \nu_p^1) \cup C(Y_{i,1}, \nu_p^2) \cup \mathrm{Preimage}(Y_{i,1}, p)$.

Let E be a component of D on $Y_{i,1}$ which contains $C(Y_{i,1}, \nu_\alpha^j)$ for some $\alpha \in T_0$ and some j. Then there exists $Y_{i,2} \to Y_{i,1}$ which is an isomorphism over

$$Y_{i,1} \setminus E \cap \left(\cup_{p' \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(Y_{i,1}, p')\right),$$

is toroidal for $\overline{D}_q$ over $W(Y_{i,1},q)\cap E$ for $q\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)$, is an isomorphism over $C(Y_{i,1},\nu_q^j)\setminus\mathrm{Preimage}(Y_{i,1},q)$ for all $q\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)$, and so that if $\overline{E}$ is the strict transform of E on $Y_{i,2}$, then for $p\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)$, one of the following holds:

$$W(Y_{i,2},p)\cap\overline{E}=\emptyset \tag{53}$$

or

(54)

There exists a unique j such that
$W(Y_{i,2},p)\cap\overline{E}\subset C(Y_{i,2},\nu_p^j)\subset\overline{E}$,
and
if $\overline{p}_j=\Lambda(Y_{i,2},\nu_p^j)$, then $C(Y_{i,2},\nu_p^j)$ is smooth at $\overline{p}_j$,
and either $\overline{p}_j$ is an isolated point in $\mathrm{Sing}_1(Y_{i,2})$ or $C(Y_{i,2},\nu_p^j)$
is the only curve in $\mathrm{Sing}_1(Y_{i,2})$ which is contained in $\overline{E}$ and contains $\overline{p}_j$,
and
$\overline{p}_j\in C(Y_{i,2},\nu_{p'}^k)$ for some $p'\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)$ implies $C(Y_{i,2},\nu_{p'}^k)=C(Y_{i,2},\nu_p^j)$.

We have that $Y_{i,2}\setminus\cup_{p\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)}W(Y_{i,2},p)$ is 2-prepared, and $\sigma_D(q)<r$ for $q\in Y_{i,2}\setminus\cup_{p\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)}W(Y_{i,2},p)$.

Now repeat this procedure for other components of D for $Y_{i,2}$ which contain a curve $C(Y_{i,2},\nu_\alpha^j)$ with $\alpha\in T_0$ for some j to construct $Y_{i,3}\to Y_{i,2}$ so that condition (53) or (54) hold for all $p\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)$ and components E of D for $Y_{i,3}$ containing a curve $C(Y_{i,3},\nu_\alpha^j)$ with $\alpha\in T_0$. We have that $Y_{i,3}\setminus\cup_{p\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)}W(Y_{i,3},p)$ is 2-prepared, and $\sigma_D(q)<r$ for $q\in Y_{i,3}\setminus\cup_{p\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)}W(Y_{i,3},p)$.

Now, by Lemma 3.4, let $Y_{i,4}\to Y_{i,3}$ be a sequence of blow ups of 2-curves of D on the strict transform of components E of D which contain $C(Y_{i,3},\nu_\alpha^j)$ for some $\alpha\in T_0$, so that if E is a component of $D_{Y_{i,4}}$ which contains a curve $C(Y_{i,4},\nu_\alpha^j)$ with $\alpha\in T_0$, and if $p\in E\setminus\cup_{q\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)}\Lambda(Y_{i,4},\nu_q^j)$ is a 2-point, then p is 3-prepared. We have that $Y_{i,4}\setminus\cup_{p\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)}W(Y_{i,4},p)$ is 2-prepared, and $\sigma_D(q)<r$ for $q\in Y_{i,4}\setminus\cup_{p\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)}W(Y_{i,4},p)$. We further have that for all $p\in T_0\cup\left(\cup_{j=1}^{i}\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\right)$, (53) or (54) holds on E.

Now let E be a component of D for $Y_{i,4}$ which contains a curve $C(Y_{i,4},\nu_\alpha^j)$ with $\alpha\in T_0$. Let

$$T=\{q\in E\mid Y_{i,4}\text{ is not 3-prepared at }q\}.$$

If $r\leq i$, let

$$T'=\left\{\begin{array}{l}\text{1-points } q \text{ of } D \text{ contained in } E \text{ such that}\\ q\in C(Y_{i,4},\nu_p^j)\text{ for some } p\in T_0 \text{ and } \sigma_D(q)>0\end{array}\right\}.$$

Since one of the conditions (53) or (54) hold for all $p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)$ on E, we may apply Proposition 3.14 to E and the finite set of points $A = T$, if $r > i$ or $A = T \cup T'$ if $r \le i$, which are necessarily 1-points for D lying on E, being sure that none of the finitely many points 2-points of D

$$B = \{\Lambda(Y_{i,4}, \nu_p^j) \mid p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)\}$$

are in the image of the general curves blown up, to construct a sequence of permissible transforms $Y_{i,5} \to Y_{i,4}$ so that $Y_{i,5} \to Y_{i,4}$ is an isomorphism in a neighborhood of $\cup_{p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(Y_{i,4}, p)$ and over $Y_{i,4} \setminus E$, and $Y_{i,5}$ is 3-prepared over

$$E \setminus \cup_{p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} \Lambda(Y_{i,4}, \nu_p^j).$$

We have that $Y_{i,5} \setminus \cup_{p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(Y_{i,5}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{i,5} \setminus \cup_{p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} W(Y_{i,5}, p)$. If $r \le i$ and $p \in T_0$, then $\sigma_D(q) = 0$ if $q \in C(Y_{i,5}, \nu_p^j)$ is a 1-point for D. We further have that for all $p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)$, (53) or (54) hold on the strict transform $\overline{E}$ of E on $Y_{i,5}$.

Now repeat this procedure for other components of $D_{Y_{i,5}}$ which contain a curve $C(Y_{i,5}, \nu_\alpha^j)$ with $\alpha \in T_0$ for some j to construct $X_{i+1,0} \to Y_{i,5}$ so that $X_{i+1,0}$ is 3-prepared over $E \setminus \cup_{p \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right)} \Lambda(Y_{i,5}, \nu_p^j)$ for all components E of D for $Y_{i,5}$ which contain a curve $C(Y_{i,5}, \nu_\alpha^j)$ with $\alpha \in T_0$.

Let

$$T_{i+1,0} = \left\{ \begin{array}{l} \text{2-points } q \text{ for } D \text{ of } C(X_{i+1,0}, \nu_p^j) \setminus \cup_{p' \in \Omega} \text{Preimage}(X_{i+1,0}, p') \\ \text{where } \Omega = \\ \{p' \in T_0 \cup \left(\cup_{j=1}^{i} \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \mid C(X_{i+1,0}, \nu_p^j) = C(X_{i+1,0}, \nu_{p'}^l) \text{ for some } l\} \\ \text{such that } \sigma_D(q) > 0 \text{ and such that } p \in T_0 \text{ with} \\ \sigma_D(\eta) = \max\{0, r-i-1\} \text{ for } \eta \in C(X_{i+1,0}, \nu_p^j) \text{ a general point.} \end{array} \right\}.$$

$X_{i+1,0}$ is 3-prepared at a point $q \in T_{i+1,0}$. For $q \in T_{i+1,0}$, choose a local resolver $(U_q, \overline{D}_q, I_q, \nu_q^1, \nu_q^2)$. Then $X_{i+1,0}$ satisfies the conclusions 2.1) - 2.6).

Case 2 Now suppose that $l < \min\{i-1, r-1\}$. We will construct $X_{i,l+1} \to X_{i,l}$. Let Ω be the set of points $q \in T_{i,l}$ such that q is contained in a curve $C(X_{i,l}, \nu_p^l)$ where $p \in T_0$ and $\sigma_D(\eta) = \max\{0, r-i\}$ for $\eta \in C(X_{i,l}, \nu_p^l)$ a general point. By condition 2.5) satisfied by $X_{i,l}$,

$$\max\{1, r-i\} \le \sigma_D(q) \le r-l-1 \tag{55}$$

for $q \in \Omega$. Let $Y \to X_{i,l}$ be a morphism which is an isomorphism over $X_{i,l} \setminus \Omega$ and is toroidal for $\overline{D}_q$ above $q \in \Omega$ and such that $C(Y, \nu_p^l) \cap W(Y, q) = \emptyset$ if $C(Y, \nu_p^l)$ is such that $p \in T_0$, $\sigma_D(\eta) = \max\{0, r-i\}$ if $\eta \in C(Y, \nu_p^l)$ is a general point, and $C(Y, \nu_p^l) \ne C(Y, \nu_q^k)$ for any k. For such a case we have by (55), that $\sigma_D(\overline{q}) \le \max\{0, r-l-2\}$ if $\overline{q} = \Lambda(Y, \nu_p^l)$. Now we may construct, using the method of Case 1, a morphism $X_{i,l+1} \to Y$ such that

$X_{i,l+1} \to X_{i,l}$ is toroidal for D above $X_{i,l} \setminus \Omega$, and the conditions 2.2) - 2.6) following (52) hold. This completes the inductive construction of (52).

For m sufficiently large in (52), we have that for $p \in T_0$, $I_p\mathcal{O}_{X_{m,r-1},\eta}$ is locally principal at a general point η of a curve $C(X_{m,r-1},\nu_p^j)$.

After possibly performing a toroidal morphism for D, we have that the locus where $I_p(\mathcal{O}_{X_{m,r-1}}|\text{Preimage}(X_{m,r-1},U_p))$ is not locally principal is supported above p for $p \in T_0$. Thus toroidal morphisms for $\overline{D}_p$ above Preimage$(X_{m,r-1},U_p)$ which principalize I_p above U_p for $p \in T_0$ extend to a morphism $Z^1 \to X_{m,r-1}$ which is an isomorphism over $X_{m,r-1} \setminus \cup_{p\in T_0}\text{Preimage}(X_{m,r-1},p)$. We have that $W(Z^1,p) = \emptyset$ for $p \in T_0$. We have that Z^1 is 2-prepared at $q \in Z^1 \setminus \cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}} W(Z^1,p)$ and $\sigma_D(q) \le r-1$ for $q \in Z^1 \setminus \cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}} W(Z^1,p)$.

If $r = 1$, then Z^1 is prepared. In this case let $X_1 = Z^1$. Suppose that $r > 1$. Let $Z_1^1 \to Z^1$ be a toroidal morphism for D so that components of D containing curves $C(Z_1^1,\nu_p^j)$ for $p \in \cup_{j=1}^m \cup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$ are pairwise disjoint, and that if $p \in \cup_{j=1}^m \cup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$, then $W(Z_1^1,p)$ is contained in $C(Z_1^1,\nu_p^1) \cup C(Z_1^1,\nu_p^2) \cup \text{Preimage}(Z_1^1,p)$.

Let E be a component of D on Z_1^1 which contains $C(Z_1^1,\nu_p^j)$ for some $p \in \cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$ or contains a point $q \in E \setminus \cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}} W(Z_1^1,p)$ such that $\sigma_D(q) = r-1$. Then there exists $Z_2^1 \to Z_1^1$ which is an isomorphism over

$$Z_1^1 \setminus E \cap (\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}} W(Z_1^1,p)),$$

is toroidal for $\overline{D}_q$ over $W(Z_1^1,q)\cap E$ for $q \in \cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$, is an isomorphism over $C(Z_1^1,\nu_q^j) \setminus \text{Preimage}(Z_1^1,q)$ for all $q \in \cup_{j=1}^m \cup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$ and factors as a sequence of permissible blow ups of points and curves

$$Z_2^1 = Z_2^{1,n} \to Z_2^{1,n-1} \to \cdots \to Z_2^{1,1} \to Z_1^1$$

such that the center blown up in $Z_2^{1,t} \to Z_2^{1,t-1}$ is a curve or point contained in $W(Z_2^{1,t-1},p)$ for some $p \in \cup_{j=1}^m \cup_{l=1}^{\min\{j-1,r-1\}} T_{j,l}$, and so that if $\overline{E}$ is the strict transform of E on Z_2^1, then for $p \in \cup_{j=1}^m \cup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$, one of the following holds:

$$W(Z_2^1,p) \cap \overline{E} = \emptyset \tag{56}$$

or

(57)

There exists a unique j such that
$W(Z_2^1,p)\cap\overline{E}\subset C(Z_2^1,\nu_p^j)\subset\overline{E}$,
and
if $\overline{p}_j=\Lambda(Z_2^1,\nu_p^j)$, then $C(Z_2^1,\nu_p^j)$ is smooth at $\overline{p}_j$,
and either $\overline{p}_j$ is an isolated point in $\mathrm{Sing}_1(Z_2^1)$ or $C(Z_2^1,\nu_p^j)$
is the only curve in $\mathrm{Sing}_1(Z_2^1)$ which is contained in $\overline{E}$ and contains $\overline{p}_j$,
and
$\overline{p}_j\in C(Z_2^1,\nu_{p'}^k)$ for some $p'\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}$ implies $C(Z_2^1,\nu_{p'}^k)=C(Z_2^1,\nu_p^j)$
and
If γ is a 2-curve of D on E which contains $\overline{p}_j$,
then $\sigma_D(q)\le r-2$ for $q\in\gamma\setminus\{\overline{p}_j\}$.

Note that no new components of D containing points

$$p\in D\setminus\left(\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}W(Z_2^1,p)\right)$$

with $\sigma_D(p)=r-1$ can be created as

$$q\in\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}(\mathrm{Preimage}(Z_2^1,W(Z_1^1,p))\setminus W(Z_2^1,p))$$

implies $\sigma_D(q)\le r-2$.

We further have that Z_2^1 is 2-prepared at $q\in Z_2^1\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}W(Z_2^1,p)$ and $\sigma_D(q)\le r-1$ for $q\in Z_2^1\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}W(Z_2^1,p)$.

Now repeat this procedure for other such components E of D for Z_2^1 which contain $C(Z_2^1,\nu_p^j)$ for some $p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}$ or contain a point

$$q\in E\setminus\cup_{p\in\cup_{j=1}^m\cup_{l=1}^{\min\{j-1,r-1\}}T_{j,k}}W(Z_2^1,p)$$

with $\sigma_D(q)=r-1$ (which are necessarily the strict transform of a component of D on Z_1^1) to construct $Z_3^1\to Z_2^1$ so that for all $p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}$, condition (56) or (57) hold for all components E of D for Z_3^1 which contain $C(Z_3^1,\nu_p^j)$ for some $p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}$ or contain a point $q\in E\setminus\cup_{p\in\cup_{j=1}^m\cup_{l=1}^{\min\{j-1,r-1\}}T_{j,k}}W(Z_3^1,p)$ with $\sigma_D(q)=r-1$. We have that Z_3^1 is 2-prepared at $q\in Z_3^1\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}W(Z_3^1,p)$ and $\sigma_D(q)\le r-1$ for $q\in Z_3^1\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}W(Z_3^1,p)$.

Now by Lemma 3.4, we can perform a torodial morphism for D (which is a sequence of blowups of 2-curves for D) $Z_4^1\to Z_3^1$, so that we further have that if G is a component of $D_{Z_4^1}$ containing a curve $C(Z_4^1,p)$ for some $p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}$ or $G\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}W(Z_4^1,p)$ contains a point q with $\sigma_D(q)=r-1$, then Z_4^1 is 3-prepared at all 2-points and 3-points of G. We further have that for all $p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}$, (56) or (57) holds on G.

We now may apply Proposition 3.14 to the union H of components E of D for Z_4^1 containing a curve $C(Z_4^1,\nu_p^j)$ for some $p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}$, or containing a point q with $\sigma_D(q)=r-1$ with

$$A=\{q\in H\mid Z_4^1 \text{ is not 3-prepared at } q \text{ (which are necessarily one points of } D)\}$$

being sure that none of the finitely many 2-points for D

$$B=\{\Lambda(Z_4^1,\nu_p^j)\mid p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}\}$$

are in the image of the general curves blown up, to construct $X^1\rightarrow Z_4^1$ so that X^1 is 3-prepared over $E\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}\Lambda(X^1,\nu_p^j)$ for all components E of D for X^1 which contain a curve $C(X^1,\nu_p^j)$ for some $p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}$, or contain a point $q\in X^1\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}W(X^1,p)$ with $\sigma_D(q)=r-1$. Further, for all $p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}$, condition (56) or (57) hold on components F of D for X^1 containing a curve $C(X^1,\nu_p^j)$ or a point $q\in X^1\setminus(\cup_{p\in\cup_{j=1}^m\cup_{k=1}^{\min\{j-1,r-1\}}T_{j,k}}W(X^1,p))$ such that $\sigma_D(q)=r-1$.

We now have (using Lemma 5.1) the following:

3.1) $X^1\rightarrow X_{j,k}$ is toroidal for $\overline{D}_p$ for $p\in T_{j,k}$ with $1\le j\le m$, $0\le k\le\min\{j-1,r-1\}$ in a neighborhood of $W(X^1,p)$.

3.2) $X^1\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}}W(X^1,p)$ is 2-prepared and $\sigma_D(q)\le r-1$ for $q\in X^1\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}}W(X^1,p)$.

3.3) Suppose that $1<r$. Then

a) X^1 is 3-prepared at all points

$$q\in C(X^1,\nu_p^k)\setminus\cup_{p'\in\cup_{j=1}^m\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}\mid C(X^1,\nu_p^j)=C(X^1,\nu_{p'}^k)\text{ for some k}}\text{Preimage}(X^1,p')$$

for $p\in\cup_{j=1}^m\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}$.

b) X^1 is 3-prepared at all points of

$$\left(X^1\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}}W(X^1,p)\right)\cap\text{Sing}_{r-1}(X^1),$$

and if $C\subset\text{Sing}_{r-1}(X^1)$ is not equal to a curve $C(X^1,\nu_p^k)$ for some $p\in\cup_{j=1}^m\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}$, then

$$C\cap\cup_{p\in\cup_{j=1}^m\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}}W(X^1,p)=\emptyset.$$

3.4) Suppose that $1<r$. Let

$$T_0^1=\left\{\begin{array}{l}\text{2-points } q \text{ of } X^1\setminus\cup_{p\in\cup_{j=1}^m\cup_{k=0}^{\min\{j-1,r-1\}}T_{j,k}}W(X^1,p)\\ \text{such that } \sigma_D(q)=r-1.\end{array}\right\}$$

For $p \in T_0^1$, let $(U_p, \overline{D}_p, \nu_p^1, \nu_p^2)$ be associated local resolvers. Let Γ_1 be the union of the curves

$$\left\{ \begin{array}{l} C(X^1, \nu_p^i) \text{ such that } p \in \left(\cup_{j=1}^m \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup T_0^1 \\ \text{and } \sigma_D(\eta) = r-1 \text{ for } \eta \in C(X^1, \nu_p^j) \text{ a general point} \end{array} \right\}$$

and any remaining curves C in

$$\mathrm{Sing}_{r-1}(X^1 \setminus \left(\cup_{j=1}^m \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup T_0^1)$$

(which are necessarily closed in X^1 and do not contain 2-points).

3.5) Suppose that $1 < r$. Suppose that

$$p \in \left(\cup_{j=1}^m \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup T_0^1$$

and $C(X^1, \nu_p^l)$ is such that $\sigma_D(\eta) = r-1$ for $\eta \in C(X^1, \nu_p^l)$ the generic point. Then $\sigma_D(q) = r-1$ for

$$q \in C(X^1, \nu_p^l) \setminus \left(\cup_{p' \in \left(\cup_{j=1}^m \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup T_0^1} W(X^1, p')\right).$$

Further, if $q \in \left(\cup_{j=1}^m \cup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}\right) \cup T_0^1$ and $W(X^1, q) \cap C(X^1, \nu_p^l) \neq \emptyset$, then $C(X^1, \nu_p^l) = C(X^1, \nu_q^n)$ for some n.

Now we proceed in this way to inductively construct sequences of blow ups for $0 \le j \le r-1$ (as in the algorithm of (52)), where we identify $X_{i,l}^0$ with $X_{i,l}$,

$$\begin{array}{l} X^j_{m_j,r-j-1} \to \cdots \to X^j_{m_j,0} \to \cdots \to X^j_{r-j,r-j-1} \to \cdots \to X^j_{r-j,0} \to X^j_{r-j-1,r-j-2} \\ \to \cdots \to X^j_{3,0} \to X^j_{2,1} \to X^j_{2,0} \to X^j_{1,0} \to X^j \end{array} \tag{58}$$

and

$$X^j \to X^{j-1}_{m_{j-1},r-j-2} \tag{59}$$

for $1 \le j \le r$ (as in the construction of X^1) such that for $1 \le j \le r$,

4.1) $X^j \to X^{j-1}_{i,k}$ is toroidal for $\overline{D}_p$ for $p \in T^{j-1}_{i,k}$ with $1 \le i \le m_{j-1}$, $0 \le k \le \min\{i-1, r-j\}$ in a neighborhood of $W(X^j, p)$.

4.2) $X^j \setminus \cup_{p \in \cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T^{j-1}_{i,k}} W(X^j, p)$ is 2-prepared and $\sigma_D(q) \le r-j$ for $q \in X^j \setminus \cup_{p \in \cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T^{j-1}_{i,k}} W(X^j, p)$.

4.3) Suppose that $j < r$. Then

a) X^j is 3-prepared at all points

$$q \in C(X^j, \nu_p^k) \setminus \cup_{p' \in \cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T^{j-1}_{i,k} | C(X^j, \nu_p^k) = C(X^j, \nu^l_{p'}) \text{ for some } l} \mathrm{Preimage}(X^j, p')$$

for $p \in \cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T^{j-1}_{i,k}$.

b) X^j is 3-prepared at all points of

$$\left(X^j \setminus \cup_{p \in \cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T^{j-1}_{i,k}} W(X^j, p)\right) \cap \mathrm{Sing}_{r-j}(X^j),$$

and if $C \subset \mathrm{Sing}_{r-j}(X^j)$ is not equal to a curve $C(X^j, \nu_p^k)$ for some $p \in \cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T_{i,k}^{j-1}$, then

$$C \cap \cup_{p \in \cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T_{i,k}^{j-1}} W(X^j, p) = \emptyset.$$

4.4) Suppose that $j < r$. Let

$$T_0^j = \left\{ \begin{array}{l} \text{2-points } q \text{ of } X^j - \cup_{p \in \cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T_{i,k}^{j-1}} W(X^j, p) \\ \text{such that } \sigma_D(q) = r - j \end{array} \right\}$$

For $p \in T_0^j$, let $(U_p, \overline{D}_p, \nu_p^1, \nu_p^2)$ be associated local resolvers.

Let Γ_j be the union of the curves

$$\left\{ \begin{array}{l} C(X^j, \nu_p^i) \text{ such that } p \in \left(\cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j \\ \text{and } \sigma_D(\eta) = r - j \text{ for } \eta \in C(X^j, \nu_p^l) \text{ a general point} \end{array} \right\}$$

and any remaining curves C in

$$\mathrm{Sing}_{r-j}(X^j \setminus \left(\cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j)$$

(which are necessarily closed in X^j and do not contain 2-points).

4.5) Suppose that $j < r$. Suppose that

$$p \in \left(\cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j$$

and $C(X^j, \nu_p^l)$ is such that $\sigma_D(\eta) = r - j$ for $\eta \in C(X^j, \nu_p^l)$ the generic point. Then $\sigma_D(q) = r - j$ for

$$q \in C(X^j, \nu_p^l) \setminus \left(\cup_{p' \in \left(\cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j} W(X^j, p') \right).$$

Further, if $q \in \left(\cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j$ and $W(X^j, q) \cap C(X^j, \nu_p^l) \neq \emptyset$, then $C(X^j, \nu_p^l) = C(X^j, \nu_q^n)$ for some n.

For $0 \le j \le r-1$, $0 \le i \le m_j$ and $0 \le k \le \min\{i-1, r-j-1\}$,

5.1) $X_{1,0}^j \to X^j$ is the canonical sequence of blow ups above a general point η of a curve in Γ_j (so that $\sigma_D(\eta) = r - j$), and for $i > 0$,

$$X_{i+1,0}^j \to X_{i,\min\{i-1,r-j-1\}}^j$$

is the canonical sequence of blow ups above a general point η of a curve

$$C(X_{i,\min\{i-1,r-j-1\}}^j, \nu_p^j)$$

with $p \in \left(\cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j$ and $\sigma_D(\eta) = \max\{0, r - i - j\}$,

and the following properties hold. Let

$$S_{i,k}^j = \left(\cup_{l=1}^{m_{j-1}} \cup_{n=0}^{\min\{l-1,r-j\}} T_{l,n}^{j-1} \right) \cup T_0^j \cup \left(\cup_{l=1}^{i-1} \cup_{n=0}^{\min\{l-1,r-j-1\}} T_{l,n}^j \right) \cup \left(\cup_{n=0}^{k-1} T_{i,n}^j \right).$$

5.2) $X^j_{i,k} \to X^s_{l,n}$ is toroidal for $\overline{D}_p$ in a neighborhood of $W(X^j_{i,k}, p)$ for $p \in S^j_{i,k}$ (with $p \in X^s_{l,n}$).

5.3) $X^j_{i,k} \setminus (\cup_{p \in S^j_{i,k}} W(X^j_{i,k}, p))$ is 2-prepared and $\sigma_D(p) < r - j$ for $q \in X^j_{i,k} \setminus (\cup_{p \in S^j_{i,k}} W(X^j_{i,k}, p))$.

5.4) If $p \in \left(\cup_{l=1}^{m_{j-1}} \cup_{n=0}^{\min\{l-1,r-j\}} T^{j-1}_{l,n}\right) \cup T^j_0$, then $\sigma_D(\eta) \leq \max\{0, r-i-j\}$ for $\eta \in C(X^j_{i,k}, \nu^l_p)$ the generic point and $X^j_{i,k}$ is 3-prepared at q for

$$q \in C(X^j_{i,k}, \nu^k_p) \setminus \cup_{p' \in S^j_{i,k} | C(X^j_{i,k}, \nu^k_p) = C(X^j_{i,k}, \nu^l_{p'}) \text{ for some } l} \mathrm{Preimage}(X^j_{i,k}, p').$$

5.5) We have the set

$$T^j_{i,k} = \left\{ \begin{array}{l} \text{2-points } q \text{ for } D \text{ of} \\ C(X^j_{i,k}, \nu^k_p) \setminus \cup_{p' \in \Omega} \mathrm{Preimage}(X^j_{i,k}, p'), \\ \text{where } \Omega = \{p' \in S^j_{i,k} \mid C(X^j_{i,k}, \nu^k_p) = C(X^j_{i,k}, \nu^l_{p'}) \text{ for some } l\} \\ \text{such that } \sigma_D(q) > 0 \text{ and such that} \\ p \in \left(\cup_{l=1}^{m_{j-1}} \cup_{n=0}^{\min\{l-1,r-j\}} T^{j-1}_{l,n}\right) \cup T^j_0 \\ \text{with } \sigma_D(\eta) = \max\{0, r-i-j\} \text{ for } \eta \in C(X^j_{i,k}, \nu^k_p) \text{ the generic point.} \end{array} \right\}$$

$X^j_{i,k}$ is 3-prepared at $p \in T^j_{i,k}$. We have local resolvers $(U_p, \overline{D}_p, I_p, \nu^1_p, \nu^2_p)$ at $p \in T^j_{i,k}$. We have $\max\{1, r-i-j\} \leq \sigma_D(q) \leq r-j-k-1$ for $q \in T^j_{i,k}$.

5.6) Suppose that

$$p \in \left(\cup_{l=1}^{m_{j-1}} \cup_{n=0}^{\min\{l-1,r-j\}} T^{j-1}_{l,n}\right) \cup T^j_0$$

and $C(X^j_{i,k}, \nu^l_p)$ is such that $\sigma_D(\eta) = \max\{0, r-i-j\}$ for $\eta \in C(X^j_{i,k}, \nu^l_p)$ a general point. Then $\sigma_D(q) = \max\{0, r-i-j\}$ for $q \in C(X^j_{i,k}, \nu^l_p) \setminus \left(\cup_{p' \in S^j_{i,k} \cup T^j_{i,k}} W(X^j_{i,k}, p')\right)$. Further,

a) If $q \in S^j_{i,k}$ and $W(X^j_{i,k}, q) \cap C(X^j_{i,k}, \nu^l_p) \neq \emptyset$, then $C(X^j_{i,k}, \nu^l_p) = C(X^j_{i,k}, \nu^n_q)$ for some n.

b) If $q \in T^j_{i,k}$ and $q \in C(X^j_{i,k}, \nu^l_p)$, then either $C(X^j_{i,k}, \nu^l_p) = C(X^j_{i,k}, \nu^n_q)$ for some n or

$$\max\{0, r-i-j\} < \sigma_D(q) \leq r-k-j-1.$$

By the definition of $T^j_{i,k}$ in 5.5) above, we have that $\cup_{i=1}^{m_{j-1}} \cup_{k=0}^{\min\{i-1,r-j\}} T^{j-1}_{i,k} = \emptyset$. Thus 4.2), following (59), implies that X^r is prepared.

□

6. Proof of Toroidalization

Theorem 6.1. *Suppose that $\mathfrak{k}$ is an algebraically closed field of characteristic zero, and $f : X \to S$ is a dominant morphism from a nonsingular 3-fold over $\mathfrak{k}$ to a nonsingular surface S over $\mathfrak{k}$ and D_S is a reduced SNC divisor on S such that $D_X = f^{-1}(D_S)_{red}$ is a SNC divisor on X which contains the locus where f is not smooth. Further suppose that f is 1-prepared. Then there exists a sequence of blow ups of points and nonsingular curves $\pi_2 : X_1 \to X$, which are contained in the preimage of D_X, such that the induced morphism $f_1 : X_1 \to S$ is prepared with respect to D_S.*

Proof. The proof is immediate from Lemma 2.2, Proposition 2.7 and Theorem 5.3. □

Theorem 6.1 is a slight restatement of Theorem 17.3 of [15]. Theorem 17.3 [15] easily follows from Lemma 2.2 and Theorem 6.1.

Theorem 6.2. *Suppose that $\mathfrak{k}$ is an algebraically closed field of characteristic zero, and $f : X \to S$ is a dominant morphism from a nonsingular 3-fold over $\mathfrak{k}$ to a nonsingular surface S over $\mathfrak{k}$ and D_S is a reduced SNC divisor on S such that $D_X = f^{-1}(D_S)_{red}$ is a SNC divisor on X which contains the locus where f is not smooth. Then there exists a sequence of blow ups of points and nonsingular curves $\pi_2 : X_1 \to X$, which are contained in the preimage of D_X, and a sequence of blow ups of points $\pi_1 : S_1 \to S$ which are in the preimage of D_S, such that the induced rational map $f_1 : X_1 \to S_1$ is a morphism which is toroidal with respect to $D_{S_1} = \pi_1^{-1}(D_S)$.*

Proof. The proof follows immediately from Theorem 6.1, and Theorems 18.19, 19.9 and 19.10 of [15]. □

Theorem 6.2 is a slight restatement of Theorem 19.11 of [13]. Theorem 19.11 [15] easily follows from Theorem 6.2.

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